Chapter 8

Asymptotic Expansion and Perturbation Theory

> *There are no solved problems; there are only problems that are more or less solved.*
>
> — Henri Poincaré

ALTHOUGH asymptotic series usually end up with a non-convergent behavior, the expansion is useful when truncated to a finite number of terms. The approximation may provide benefits by being more mathematically tractable or by being more computationally friendly than the expanded function.

8.1 Order Notations

The $\mathcal{O}$ (big O) and o (little O) order notations (introduced by Landau) are often used for comparison of function limits.

1. *Asymptotically bounded.*

   $$f(z) = \mathcal{O}(g(z)), \qquad z \to z_0,$$

   means that there exists a constant $C(\delta) > 0$ that

   $$\left|\frac{f(z)}{g(z)}\right| \le C, \quad |z - z_0| < \delta.$$

2. *Asymptotically negligible.*

   $$f(z) = o(g(z)), \qquad z \to z_0,$$

   means that for every $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$ that

   $$\left|\frac{f(z)}{g(z)}\right| \le \varepsilon, \quad |z - z_0| < \delta.$$

3. *Asymptotically equal.*

   $$f(z) \sim g(z), \quad z \to z_0,$$

   means that $|f(z)/g(z)| \to 1$ as $z \to z_0$.

Example 8.1.1.

1. $\sin(x^{-10}) = \mathcal{O}(1)$ *as* $x \to 0$*, but not the other way.*
2. $x = o(\ln x)$ *as* $x \to 0^+$*.*
3. $x \sim \tan(x)$ *as* $x \to 0^+$*.*
4. $e^{-1/\sqrt{x}} = o(x^n)$ *as* $x \to 0^+$ *for any* $n \in \mathbb{N}$*.*

However, the Big-O and little-O notations are **not** quantitative without estimates for the constants $C(\delta)$ or $\delta(\varepsilon)$.

8.2 Asymptotic Expansion

In a nutshell, the *asymptotic expansion* describing a limiting procedure for some function $f(z)$ with respect to a certain parameter $|z - z_0| \ll 1$, which results in a series form: $\sum_{k=1}^{\infty} \phi_k(z)$, such that the successive terms are *much smaller* than the previous ones when the parameter $z \to z_0$, intuitively, a truncation of the series will provide a reasonable approximation for f (although the series might diverge).

Definition 8.2.1. *A sequence of functions $\phi_k : \mathbb{C} \setminus \{z_0\} \to \mathbb{C}$ is an **asymptotic sequence** as $z \to z_0$ if for each $k \in \mathbb{N}$,*

$$\phi_{k+1}(z) = o(\phi_k(z)), \quad z \to z_0.$$

*These functions are called **gauge functions**.*

Definition 8.2.2. *If $\{\phi_k\}$ is an asymptotic sequence and $f : \mathbb{C}\setminus\{z_0\} \to$ **b** satisfies that*

$$f(z) - \sum_{k=0}^{N} a_k \phi_k(z) = o(\phi_N(z)) \quad z \to z_0.$$

*Then we write $f(z) \sim \sum_{k=0}^{\infty} a_k \phi_k(z)$ and call the series as the **asymptotic expansion** for f. The coefficients can be determined by*

$$a_n = \lim_{z \to z_0} \frac{f(z) - f_{n-1}(z)}{\phi_n(z)}, \quad f_{-1} \equiv 0.$$

Example 8.2.3. *The sequence $\log x$, $x \log x$, x, $x^2 \log^2 x$, $x^2 \log x$, x^2, ... is an asymptotic sequence when $x \to 0^+$.*

Example 8.2.4. *The asymptotic expansion of $e^{-1/z}$ with sequence $\phi_k = z^k$, as $z \to 0^+$, then all coefficients $a_n = 0$. Because the function $e^{-1/z} = o(z^n)$ for every $n \in \mathbb{N}$, it is called **transcendentally small**.*

Moreover, adding a transcendentally small function will not change the asymptotic expansion. For instance,

$$\frac{1}{1-z} \sim 1 + z + z^2 + \cdots, \quad z \to 0^+.$$

and

$$e^{-1/z} \sin e^{1/z} + \frac{1}{1-x} \sim 1 + z + z^2 + \cdots, \quad z \to 0^+.$$

However, the relation $\sim$ cannot be carried to term-wise differentiation; that is, a transcendentally small function may have a nonzero asymptotic expansion. For example, the above function's derivative has an asymptotic expansion

$$\frac{\cos e^{1/z}}{z^2} + 1 + 2z + 3z^2 + \cdots$$

Example 8.2.5 (Asymptotic power series)**.** *The monomial* $\phi_k(z) = z^k$ *is the most common sequence for asymptotic analysis around the origin. If* $f \in C^\infty(\mathbb{R})$ *(in* $\mathbb{C}$*, this means holomorphic) in a neighborhood of the origin, then the Taylor series*

$$\left| f(z) - \sum_{k=0}^{N} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k \right| \leq C_{N+1}|z-z_0|^{N+1}, \qquad |z-z_0| \leq \varepsilon.$$

where

$$C_{N+1} = \sup_{|z-z_0|\leq\varepsilon} \frac{|f^{(N+1)}(z)|}{(N+1)!}.$$

Therefore, the asymptotic power series of f *is*

$$f \sim \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k, \quad z \to z_0.$$

The asymptotic power series may not converge to f *or even diverge in the neighborhood of* z_0*. The bump function*

$$\psi(x) = \begin{cases} \exp(-\frac{1}{1-x^2}), & x \in (-1,1) \\ 0, & |x| \geq 1 \end{cases}$$

belongs to C^∞*. But its asymptotic series is identically zero as* $x \to \pm 1$*.*

The following theorem shows that the asymptotic power series does not need any growth rate for the coefficients.

Theorem 8.2.6 (Borel-Ritt Theorem)**.** *Given any sequence* $\{a_n\}$ *of real (or complex) coefficients, there exists a* C^∞ *function* f *such that*

$$f(z) \sim \sum_{n=0}^{\infty} a_n(z-z_0)^n, \quad z \to z_0.$$

Proof Sketch. The key idea is to make each summand $a_n(z-z_0)^n$ with shrinking support so that the summation will not *blow up* and the behavior near z_0 is still reserved.

Choose the bump-shape function $\psi \in C^\infty$ that $\psi(x) = 1$ if $|x| < 1$ and $\psi = 0$

if $|x| > 2$. Then take appropriate parameters $r_n \to 0$ that

$$|a_n| \|(z-z_0)^n \psi(\frac{z-z_0}{r_n})\|_{C^n} \leq \frac{1}{2^n}.$$

Then $f = \sum a_n (z-z_0)^n \psi(\frac{z-z_0}{r_n})$ converges pointwise. For any fixed z near z_0, only a finite number of terms are involved, and the series converges in C^n norm for every $n \in \mathbb{N}$. ☐

We use the following example to demonstrate the key difference between a convergent series and an asymptotic series.

Example 8.2.7. *Let* $\mathrm{E}_1(x)$ *be the* ***exponential integral***

$$\mathrm{E}_1(x) = \int_x^\infty \frac{e^{-t}}{t} dt.$$

And we consider the asymptotic expansion for $x \gg 1$. *Using integration by parts,*

$$\begin{aligned} \mathrm{E}_1(x) &= \left[\frac{-e^{-t}}{t}\right]\Bigg|_x^\infty - \int_x^\infty \frac{e^{-t}}{t^2} dt \\ &= \frac{e^{-x}}{x} + \left[\frac{e^{-t}}{t^2}\right]\Bigg|_x^\infty + 2\int_x^\infty \frac{e^{-t}}{t^3} dt \\ &= \frac{e^{-x}}{x} + \frac{e^{-x}}{x^2} + 2\int_x^\infty \frac{e^{-t}}{t^3} dt \\ &= \cdots \\ &= \underbrace{e^{-x} \sum_{k=1}^N (-1)^{k-1} \frac{(k-1)!}{x^k}}_{=S_N(x)} + (-1)^N N! \int_x^\infty \frac{e^{-t}}{t^{N+1}} dt. \end{aligned}$$

- *For any fixed* x, *the series* $S_N(x)$ *will diverge as* $N \to \infty$; *the radius of convergence is zero.*
- *For any fixed* N *and let* $x \to \infty$, *we can estimate the remainder*

$$\left|(-1)^N N! \int_x^\infty \frac{e^{-t}}{t^{N+1}} dt\right| \leq \frac{N!}{x^{N+1}} e^{-x},$$

which converges to zero as $x \to \infty$ *(asymptotically negligible compared with* e^{-x}/x^N*). The resulting series is asymptotic in the sense that*

$$\mathrm{E}_1(x) = S_N(x) + o(e^{-x}/x^N).$$

The key point is, for x *sufficiently large,* $S_N(x)$ *will first decrease as* N *increases, then af-*

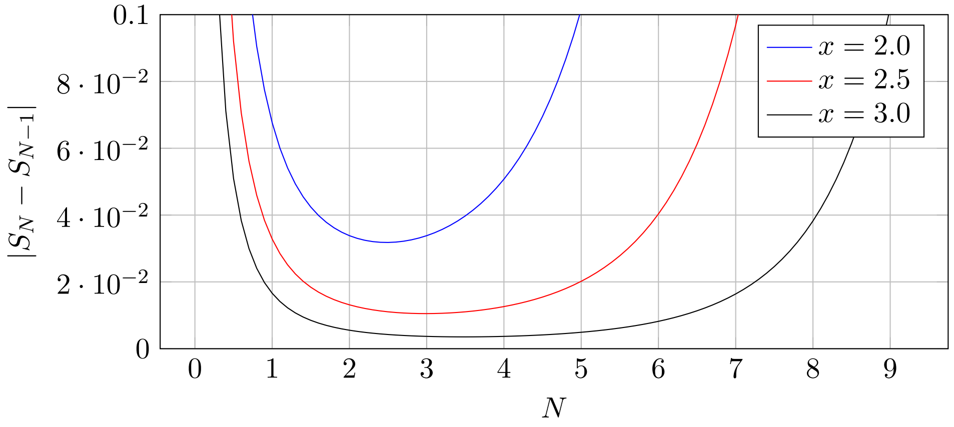

Figure 8.1: Plot of $|S_N - S_{N-1}|$.

ter a certain N^**,* S_N *will start to increase (in magnitude), see Figure 8.1. A rough estimate of* N^* *is the largest* N *that makes remainder bounded by the last term in* S_N*:*

$$\frac{N!}{x^{N+1}} e^{-x} \leq \frac{(N-1)!}{x^N} e^{-x},$$

that is $N \leq x$*.*

Remark 8.2.8. *Although the asymptotic series diverges for large* N*, the estimate with truncation at a small* N *is already reasonably accurate. For instance,* $\mathrm{E}_1(10) \approx 4.157 \times 10^{-6}$ *and* $S_4(10) \approx 4.150 \times 10^{-6}$*.*

Example 8.2.9 (Incomplete Gamma Function)**.** *The incomplete Gamma function is defined by*

$$\Gamma(a, x) = \int_0^x e^{-t} t^{a-1} dt.$$

For large x*,* $\Gamma(a,x) = \Gamma(a) - \displaystyle\int_x^\infty e^{-t} t^{a-1} dt$*. Using integration by parts, we obtain*

$$\Gamma(a, x) = \sum_{k=1}^{N} \frac{\Gamma(a)}{\Gamma(a-k+1)} e^{-x} x^{a-k} + \frac{\Gamma(a)}{\Gamma(a-N)} \Gamma(a-N, x).$$

Following a similar argument to the previous example, the asymptotic series is ($a \notin \mathbb{Z}$*).*

$$\Gamma(a, x) \sim \sum_{k=1}^{\infty} \frac{\Gamma(a)}{\Gamma(a-k+1)} e^{-x} x^{a-k}.$$

8.2.1 Laplace's Method

Laplace's method is used to estimate integrals like

$$\mathcal{I}(\lambda) = \int_a^b f(x) e^{\lambda\phi(x)} dx$$

where $f(x)$ and $\phi(x)$ are real, continuous functions. Such an integral is called *Laplace Integral*. Typically, λ is large and we are only interested in an approximation instead of an analytic formulation (if it exists).

• When ϕ is monotone and f does not vanish at two endpoints, we can rewrite the integral using integration by parts,

$$\begin{aligned}
\mathcal{I}(\lambda) &= \frac{1}{\lambda}\int_a^b \frac{f(x)}{\phi'(x)} \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{\lambda\phi(x)}\right) dx \\
&= \left[\frac{1}{\lambda}\frac{f(x)}{\phi'(x)} e^{\lambda\phi(x)}\right]\Bigg|_a^b - \frac{1}{\lambda}\int_a^b \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{\phi'(x)}\right) e^{\lambda\phi(x)} dx \\
&= \left[\frac{1}{\lambda}\frac{f(x)}{\phi'(x)} e^{\lambda\phi(x)}\right]\Bigg|_a^b - \frac{1}{\lambda^2}\left[\frac{1}{\phi'(x)}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{\phi'(x)}\right) e^{\lambda\phi(x)}\right]\Bigg|_a^b \\
&\quad + \frac{1}{\lambda^2}\int_a^b \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\phi'(x)}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{\phi'(x)}\right)\right) e^{\lambda\phi(x)} dx.
\end{aligned}$$

The first term dominates as $\lambda \to \infty$ (why?), which means

$$\mathcal{I}(\lambda) \sim \frac{1}{\lambda}\left[\frac{f(b)}{\phi'(b)} e^{\lambda\phi(b)} - \frac{f(a)}{\phi'(a)} e^{\lambda\phi(a)}\right], \quad \lambda \to \infty.$$

Therefore, the boundary part contributes the most.

• Suppose ϕ has a (unique) global maximum at $x = x^* \in (a, b)$ and $f(x^*) \neq 0$, we will see that the neighborhood around x^* (not the boundary) contributes the most to the integral. Decompose the integral into three regions,

$$\int_a^b f(x) e^{\lambda\phi(x)} dx = \left(\int_a^{x^*-\varepsilon} + \int_{x^*-\varepsilon}^{x^*+\varepsilon} + \int_{x^*+\varepsilon}^b\right) f(x) e^{\lambda\phi(x)} dx,$$

where $\varepsilon > 0$ is any positive number. We first claim that

$$\left|\int_a^{x^*-\varepsilon} f(x) e^{\lambda\phi(x)} dx\right| + \left|\int_{x^*+\varepsilon}^b f(x) e^{\lambda\phi(x)} dx\right|$$

will be bounded by $\mathcal{O}(e^{\lambda(\phi(x^*-\delta)})$ for a certain $\delta > 0$.

On the neighborhood of x^*, we take the Taylor series

$$\phi(x) = \phi(x^*) + \frac{1}{2}\phi''(x^*)(x - x^*)^2 + O(|x - x^*|^3).$$

The integral ($\lambda \gg \varepsilon^{-2}$)

$$\begin{aligned}
\int_{x^*-\varepsilon}^{x^*+\varepsilon} f(x)e^{\lambda\phi(x)}dx &= f(x')\int_{-\varepsilon}^{\varepsilon} e^{\lambda(\phi(x^*)+\frac{1}{2}\phi''(x^*)u^2+O(|u|^3))}du \\
&= f(x')e^{\lambda\phi(x^*)}\int_{-\varepsilon}^{\varepsilon} e^{\lambda(\frac{1}{2}\phi''(x^*)u^2+O(|u|^3))}du \\
&= \frac{f(x')e^{\lambda\phi(x^*)}}{\sqrt{\lambda|\phi''(x^*)|}}\int_{-\sqrt{\lambda|\phi''(x^*)|}\varepsilon}^{\sqrt{\lambda|\phi''(x^*)|}\varepsilon} e^{-\frac{1}{2}y^2+O(\lambda^{-1/2}|y|^3)}dy \\
&\approx \frac{f(x')e^{\lambda\phi(x^*)}}{\sqrt{\lambda|\phi''(x^*)|}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2}(1+O(\lambda^{-1/2}|y|^3))dy \\
&\sim \frac{f(x')e^{\lambda\phi(x^*)}}{\sqrt{\lambda|\phi''(x^*)|}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2}dy \\
&= \frac{\sqrt{2\pi}f(x')e^{\lambda\phi(x^*)}}{\sqrt{\lambda|\phi''(x^*)|}}.
\end{aligned}$$

where $x' \in (x^* - \varepsilon, x^* + \varepsilon)$ from mean value theorem. Shrinking $\varepsilon \to 0^+$ (and $\lambda \gg \varepsilon^{-2}$), we obtain the asymptotic approximation

$$\int_{x^*-\varepsilon}^{x^*+\varepsilon} f(x)e^{\lambda\phi(x)} \sim \frac{\sqrt{2\pi}f(x^*)e^{\lambda\phi(x^*)}}{\sqrt{\lambda|\phi''(x^*)|}}, \qquad \lambda \to \infty.$$

The asymptotic approximation only depends on the local information at x^*. Thus, if ϕ attains a global maximum at multiple isolated points (and f does not vanish on them), the result will be the sum of local approximations.

Remark 8.2.10. *If $\phi^{(k)}(x^*) = 0$ for $k = 1, 2\cdots, m$, then local approximation still applies*

$$\phi(x) = \phi(x^*) + \frac{\phi^{(m+1)}(x^*)}{(m+1)!}(x - x^*)^{m+1} + O(|x - x^*|^{m+2}).$$

The result will be replaced by

$$2\Gamma\left(\frac{m+2}{m+1}\right) f(x^*)e^{\lambda\phi(x^*)}\left(\frac{(m+1)!}{\lambda|\phi^{(m+1)}(x^*)|}\right)^{1/(m+1)}.$$

Remark 8.2.11. *If $f^{(2k)}(x^*) = 0$ for all $0 \le k \le n-1$, then locally we can replace $f(x) = \frac{f^{(2n)}(x')}{(2n)!}(x - x^*)^{2n}$, where $x' \in (x^* - \varepsilon, x^* + \varepsilon)$. The derivation will be quite* *similar.*

Example 8.2.12 (Stirling's formula). *In Chapter 7, we have used Laplace's method to derive* $\Gamma(x) \sim \sqrt{2\pi} e^{-x} x^{x-1/2}$ *as* $x \to \infty$*. This can be continued to high orders*

$$\Gamma(x) \sim \left(\frac{2\pi}{x}\right)^{1/2} x^x e^{-x} \left[1 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots\right]$$

To determine the coefficients a_i*, we consider*

$$\Gamma(z) = e^{-z} z^z \int_0^\infty e^{-zf(u)} du, \qquad f(u) = u - \ln u - 1.$$

The only global maximum of $f(u)$ *is zero at* $u = 1$*. We only focus on the integral around* $u = 1$*.*

- *For* $u \in [1, 1+\varepsilon]$*, Let*

$$t := f(u) \sim \sum_{s=2}^{\infty} \frac{1}{s}(-1)^s (u-1)^s \approx \frac{1}{2}(u-1)^2 - \frac{1}{3}(u-1)^3 + \cdots$$

and we solve $u - 1$ *in terms of* t*. Then use the following change of variable to calculate the asymptotic expansion around* $u = 1$*.*

$$\int_1^{1+\varepsilon} e^{-zf(u)} du = \int_0^{f(1+\varepsilon)} e^{-zt} \frac{du}{dt} dt = \int_0^{f(1+\varepsilon)} e^{-zt} \frac{1}{f'(f^{-1}(t))} dt$$

We use `Mathematica` *to perform the calculations.*

```
>> u = InverseSeries[Series[t - Log[t] - 1, {t, 1, 6}]]
```

and it produces the inverse (truncated) series

$$u = 1 + \sqrt{2}\sqrt{t} + \frac{2t}{3} + \frac{t^{3/2}}{9\sqrt{2}} - \frac{2t^2}{135} + \frac{t^{5/2}}{540\sqrt{2}} + O\left(t^3\right)$$

and $f'(u) = \frac{u-1}{u}$*, and use*

```
>> Simplify[1/Simplify[1 - 1/u]]
```

we obtain

$$(f'(f^{-1}(t)))^{-1} = \frac{1}{\sqrt{2}\sqrt{t}} + \frac{2}{3} + \frac{\sqrt{t}}{6\sqrt{2}} - \frac{4t}{135} + \frac{t^{3/2}}{216\sqrt{2}} + O\left(t^2\right).$$

- *For* $u \in [1-\varepsilon, 1]$*, we first find the correct series by a reflection of the variable,*

$$s = g(v) := f(-v) = -v - \ln(-v) - 1 = \sum_{s=2}^{\infty} \frac{1}{s}(1+v)^s$$

and

$$\begin{aligned}\int_{1-\varepsilon}^{1} e^{-zf(u)} du &= \int_{-1}^{-1+\varepsilon} e^{-zg(v)} dv = \int_{0}^{g(-1+\varepsilon)} e^{-zs} \frac{dv}{ds} ds \\ &= \int_{0}^{g(-1+\varepsilon)} e^{-zs} \frac{1}{g'(g^{-1}(s))} ds.\end{aligned}$$

```
>> v = InverseSeries[Series[-s - Log[-s] - 1, {s, -1, 10}]]
```

It produces

$$v = -1 + \sqrt{2}\sqrt{s} - \frac{2s}{3} + \frac{s^{3/2}}{9\sqrt{2}} + \frac{2s^2}{135} + \frac{s^{5/2}}{540\sqrt{2}} + O\left(s^3\right).$$

Then follow the same steps, $\frac{\mathrm{d}}{\mathrm{d}v} g(v) = -1 - \frac{1}{v}$*,*

```
>> Simplify[1/Simplify[-1 - 1/v]]
```

We obtain

$$(g'(g^{-1}(s)))^{-1} = \frac{1}{\sqrt{2}\sqrt{s}} - \frac{2}{3} + \frac{\sqrt{s}}{6\sqrt{2}} + \frac{4s}{135} + \frac{s^{3/2}}{216\sqrt{2}} + O\left(s^2\right).$$

Therefore, using the following fact (see next theorem),

$$\int_0^{\delta} e^{-zt} t^{n/\mu} dt = z^{-n/\mu} \int_0^{z\delta} e^{-u} u^{n/\mu} dt \xrightarrow{z \gg 1} z^{-n/\mu} \Gamma(n/\mu + 1),$$

We find that

$$\begin{aligned} z^{-z}e^{z}\Gamma(z) &\sim \frac{2}{\sqrt{2}}z^{1/2}\Gamma\left(\frac{1}{2}\right)+\frac{2}{6\sqrt{2}}z^{-1/2}\Gamma\left(\frac{3}{2}\right)+\frac{2}{216\sqrt{2}}z^{-3/2}\Gamma\left(\frac{5}{2}\right)+\cdots \\ &= \sqrt{2\pi}z^{1/2}+\frac{\sqrt{\pi}}{6\sqrt{2}}z^{-1/2}+\frac{\sqrt{\pi}}{144\sqrt{2}}z^{-3/2}+\cdots \end{aligned}$$

Thus,

$$\Gamma(z) \sim \sqrt{2\pi}z^{z+1/2}e^{-z}\left[1+\frac{1}{12}\frac{1}{z}+\frac{1}{288}\frac{1}{z^2}+\cdots\right].$$

In the above example, we have used the following result.

Theorem 8.2.13 (Watson's Lemma)**.** *Suppose* $\phi(t)$ *is analytic in the sector* $0<|t|<R$, $|\arg t|\le\delta<\pi$, *and suppose*

$$\phi(t)=\sum_{k=1}^{\infty}a_k t^{k/n-1}, \quad 0<|t|<R.$$

and $|\phi(t)|\le Ce^{bt}$ *for* $R\le z\le T$*, then*

$$\int_0^T e^{-zt}\phi(t)dt \sim \sum_{k=1}^{\infty}\frac{a_k\Gamma(k/n)}{z^{k/n}},$$

as $|z|\to\infty$ *in the sector* $|\arg z|\le\delta<\pi/2$ *(right half-plane).*

Proof Sketch. The integral

$$\int_0^T e^{-zt}\phi(t)dt=\int_0^R e^{-zt}\phi(t)dt+\int_R^T e^{-zt}\phi(t)dt,$$

and the tail

$$\left|\int_R^T e^{-zt}\phi(t)dt\right|\le\int_R^T e^{-\Re zt}Ce^{bt}dt\le C\int_R^T e^{-(\Re z-b)t}dt\le C'e^{-(\Re z-b)R},$$

which is transcendentally small as $|z|$ is sufficiently large. On $0<t<R$, because ϕ is analytic, the expansion of ϕ converges absolutely (treat $t^{1/n}$ as a variable), then use the remainder formula of the Taylor series,

$$\phi(t)=\sum_{k=1}^{m-1}a_k t^{k/n-1}+\mathcal{O}(t^{m/n-1}).$$

It only remains to see whether the remainder contributes much smaller, which is straightforward since

$$\int_0^\infty e^{-zt} t^{m/n-1} dt = \Gamma(m/n) z^{-m/n} = o(z^{k/n}), \quad k < m.$$

□

In general, the asymptotic expansion of high order can be derived (but tediously) in the same way as the previous example. We refer the interested readers to Wong (2001). The following examples are direct applications of Watson's Lemma.

Example 8.2.14. *Let*

$$f(z) = \int_0^\infty \frac{e^{-t}}{t+z} dt$$

Then we set $t = |z|s$ *(to apply Watson's Lemma),*

$$\begin{aligned} f(z) &\sim \frac{|z|}{z} \int_0^\infty \frac{e^{-|z|s}}{1 + \frac{|z|s}{z}} ds \\ &= \frac{|z|}{z} \int_0^\infty e^{-|z|s} \sum_{k=0}^\infty (-1)^k \left(\frac{|z|s}{z}\right)^k ds \\ &= \sum_{k=0}^\infty (-1)^k \left(\frac{1}{z}\right)^{k+1} \Gamma(k+1). \end{aligned}$$

Example 8.2.15 (Modified Bessel Function). *The modified Bessel function of the second kind of order zero*

$$\begin{aligned} K_0(x) &= \int_1^\infty (s^2-1)^{-1/2} e^{-xs} ds \\ &\overset{t=s-1}{=\!=} e^{-x} \int_0^\infty e^{-xt} (t^2+2t)^{-1/2} dt \\ &= e^{-x} \int_0^\infty e^{-xt} (2t)^{-1/2} (1 + {}^t\!/\!_2)^{-1/2} dt \\ &= e^{-x} \int_0^\infty e^{-xt} (2t)^{-1/2} \sum_{k=0}^\infty (-t/2)^k \frac{\Gamma(k + {}^1\!/\!_2)}{k!\Gamma({}^1\!/\!_2)} dt \\ &= e^{-x} \sum_{k=0}^\infty (-1)^k \frac{\Gamma(k+{}^1\!/\!_2)^2}{2^{k+1/2} k! \Gamma({}^1\!/\!_2)} \frac{1}{x^{k+1/2}}. \end{aligned}$$

8.2.2 Steepest Descent Method

The **steepest descent method** (do not confuse with the optimization method) originated from Riemann, which is a classical approach for estimating contour integrals like

$$\int_\gamma f(z)e^{\lambda h(z)}dz, \quad \lambda \to \infty,$$

where f and h are holomorphic functions on an open connected set that contains γ. Let $h(z) = \psi(z) + i\phi(z)$; both are harmonic, thus there are no local maxima, and the critical points are saddle points. The steepest descent method selects a contour γ that passes through a critical point such that $\psi(z)$ increases or decreases rapidly on γ. Because $\nabla\phi \cdot \nabla\psi = 0$ (see Remark 0.4.9), if we choose γ so that ψ follows a steepest descent path (orthogonal to all levelsets), then $\phi \equiv C$ on γ.

Example 8.2.16 (Airy Function)**.** *The airy function*

$$\begin{aligned}\mathrm{Ai}(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(zx+z^3/3)}dz \\ &\overset{z=x^{1/2}t}{=\!=\!=} \frac{x^{1/2}}{2\pi}\int_{-\infty}^{\infty} e^{ix^{3/2}(t+t^3/3)}dt\end{aligned}$$

Therefore, we only deal with

$$\mathcal{I}(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda(t+t^3/3)}dt$$

The function $h(t) = i(t + t^3/3) = \psi + i\phi$ has critical points at $t = \pm i$, and the real/imaginary parts are

$$\begin{aligned}\psi(x, y) &= -y(1 + x^2 - \frac{1}{3}y^2), \\ \phi(x, y) &= x(1 + \frac{1}{3}x^2 - y^2).\end{aligned}$$

Through $z = i$ or $(0, 1)$, then $\phi(0, 1) = 0$, which means the path γ is (see Figure 8.2)

$$y = \sqrt{1 + \frac{1}{3}x^2}.$$

It is straightforward to show that

$$\mathcal{I}(\lambda) = \int_\gamma e^{i\lambda(z+z^3/3)}dz.$$

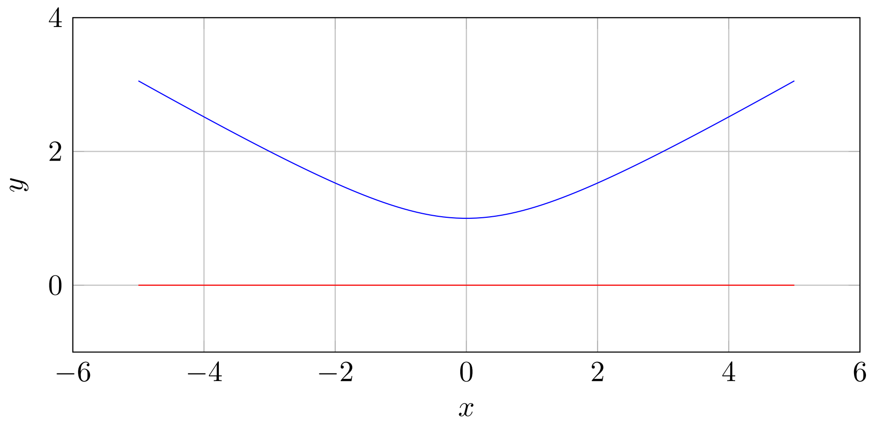

Figure 8.2: Steepest descent path (blue).

Since the real part $|e^{ih(z)}| = e^{-y(1+x^2-\frac{1}{3}y^3)}$ *decays fast as* $x \to \infty$ *between the two curves.*

On the path γ*, we reparametrize for convenience* $z(t) = \sqrt{3}\sinh(t) + i\cosh(t)$*,*

$$\mathcal{I}(\lambda) = \int_{-\infty}^{\infty} f(t)e^{\lambda q(t)}dt,$$

where $f(t) = \sqrt{3}\cosh(t) + i\sinh(t)$ *and* $q(t) = \cosh(t)[2 - {}^{8}\!/_{3}\cosh^2 t]$*. The rest will follow Laplace's method.*

We calculate $q(t)$ *attains maximum at* $t = 0$*,* $q(0) = -2/3$*,* $q''(0) = -6$*. Thus, the asymptotic approximation is*

$$\mathcal{I}(\lambda) \sim \sqrt{\frac{2\pi}{-\lambda q''(0)}} f(0)e^{\lambda q(0)} = \sqrt{\frac{\pi}{\lambda}}e^{-2\lambda/3}.$$

Thus, $\mathrm{Ai}(x) \sim \frac{1}{2\sqrt{\pi}x^{1/4}}e^{-2x^{3/2}/3}$*, as* $x \to \infty$*. When* $x \to -\infty$*, we will cover this case in the next part.*

Example 8.2.17. *Consider the integral*

$$f(x) = \frac{1}{2\pi i}\int_C e^{x(z-z^{1/2})}\frac{dz}{z}.$$

where C *is a vertical line* $z = 1 + iy$*,* $y \in (-\infty, \infty)$*.*

We first find the saddle point for $h(z) = z - z^{\mu}$ *by solving* $1 = \frac{1}{2}z^{-1/2}$*, the saddle point is* $z_0 = \frac{1}{4}$*. At* z_0*,* $h(z_0) = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4} \in \mathbb{R}$*, therefore, the contour is* $\Im h(z) = 0$*.*

Let $z = \rho e^{i\theta}$, then

$$\rho \sin(\theta) - \rho^{1/2} \sin(\frac{1}{2}\theta) = 0 \to \rho = \left(\frac{\sin(\frac{1}{2}\theta)}{\sin\theta}\right)^2 = \frac{1}{2(1+\cos\theta)}$$

The plot of the curve is a parabola $x = \frac{1}{4} - y^2$. We parametrize the curve by the y variable directly,

$$h(z) = (\frac{1}{4} - y^2 + iy) - \sqrt{\frac{1}{4} - y^2 + iy} = \frac{1}{4} - y^2 + iy - (\frac{1}{2} + iy) = -\frac{1}{4} - y^2.$$

The integral becomes

$$\begin{aligned} f(x) &= \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-x(\frac{1}{4}+y^2)} \frac{1}{(\frac{1}{2}+iy)} dy \\ &= \frac{1}{\pi} e^{-x} \int_{-\infty}^{\infty} e^{-xy^2} \frac{1}{(\frac{1}{2}+iy)} dy \\ &\sim \frac{2}{\pi} e^{-x} \int_{-\infty}^{\infty} e^{-xy^2} \sum_{k=0}^{\infty} (-1)^k 2^{2k} y^{2k} dy \\ &\sim \frac{1}{\pi} e^{-x} \int_{0}^{\infty} e^{-xt} \sum_{k=0}^{\infty} (-1)^k 2^{2k} t^{k-1/2} dt \\ &\sim \frac{2}{\pi} e^{-x} \sum_{k=0}^{\infty} (-1)^k 2^{2k} \Gamma(k+1/2) x^{-k-1/2}. \end{aligned}$$

Remark 8.2.18. *The typical steepest descent method can be summarized as follows. Consider*

$$\mathcal{I}(\lambda) = \int_\gamma f(z) e^{\lambda h(z)} dz$$

At the saddle point z_0,

$$h(z) = h(z_0) + \frac{1}{2} h''(z_0)(z-z_0)^2 + \mathcal{O}((z-z_0)^3).$$

The steepest descent path then becomes (imaginary part cancels)

$$h(z) = h(z_0) - \tau^2, \quad -\infty < \tau < \infty$$

which determines $z = z(\tau)$ and

$$\mathcal{I}(\lambda) \sim e^{\lambda h(z_0)} \int_{-\infty}^{\infty} f(z(\tau)) e^{-\lambda\tau^2} \frac{dz}{d\tau} d\tau.$$

The leading expansion of $f(z(\tau))$ *comes from*

$$f(z(\tau)) = f(z_0) + f'(z_0)(z(\tau) - z_0) + \cdots$$

and $z - z_0 = \sqrt{\frac{-2}{h''(z_0)}}\tau + \mathcal{O}(\tau^2)$. *Thus, we can find the leading term*

$$\mathcal{I}(\lambda) \sim e^{\lambda h(z_0)} f(z_0) \sqrt{\frac{-2\pi}{h''(z_0)}}.$$

Here $h''(z_0)$ *is a complex number,* $\sqrt{\frac{-2}{h''(z_0)}}$ *needs to select the suitable branch such that the direction of the path is compatible with the integral.*

8.2.3 Stationary Phase Theory

The stationary phase method deals with the special case that

$$\mathcal{I}(\lambda) = \int_a^b f(t) e^{i\lambda\phi(t)} dt,$$

where $f(t)$ and $\phi(t)$ are both real valued. Thus, we cannot exploit the exponential decay like we did in Laplace's method. However, the rough idea is the same.

- If ϕ is monotone, the integration by parts still produces an estimate

$$\mathcal{I}(\lambda) \sim \left[\frac{f(t)}{i\lambda\phi'(t)} e^{i\lambda\phi(t)}\right]\Bigg|_a^b \underbrace{- \frac{1}{i\lambda} \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{f(t)}{\phi'(t)}\right) e^{i\lambda\phi(t)} dt}_{=o(\lambda^{-1})}.$$

The second term's integral $\to 0$ due to the Riemann-Lebesgue lemma. Thus, the second term $= o\left(\frac{1}{\lambda}\right)$.

- If $\phi'(c) = 0$ for a unique stationary point $c \in (a, b)$, the integration by parts fails. Following Laplace's method, we can decompose the integral into

$$\mathcal{I}(\lambda) = \left(\int_a^{c-\varepsilon} + \int_{c-\varepsilon}^{c+\varepsilon} + \int_{c+\varepsilon}^b\right) f(t) e^{i\lambda\phi(t)} dt,$$

where $\varepsilon = \lambda^{-\delta}$ for a tiny $\delta > 0$. On the regions away from c, we can use integration by parts to obtain $\mathcal{O}(\lambda^{-(1-\delta)})$ behavior. Near the stationary point c,

$$\phi(t) = \phi(c) + \frac{1}{2}\phi''(c)(t - c)^2 + O(|t - c|^3),$$

and

$$\begin{aligned}
&\int_{c-\varepsilon}^{c+\varepsilon} f(t)e^{i\lambda(\phi(c)+\frac{1}{2}\phi''(c)(t-c)^2+O(|t-c|^3))}dt \\
&\sim f(c)e^{i\lambda\phi(c)}\sqrt{\frac{2}{\lambda|\phi''(c)|}}\int_{-\sqrt{\frac{\lambda|\phi''(c)|}{2}}\varepsilon}^{\sqrt{\frac{\lambda|\phi''(c)|}{2}}\varepsilon} e^{i\,\mathrm{sgn}(\phi''(c))t^2}(1+O(s^3/\lambda^{1/2}))ds \\
&\sim f(c)e^{i\lambda\phi(c)}\sqrt{\frac{2}{\lambda|\phi''(c)|}}\int_{-\infty}^{\infty} e^{i\,\mathrm{sgn}(\phi''(c))s^2}ds \\
&= f(c)\sqrt{\frac{2\pi}{\lambda|\phi''(c)|}}e^{i\,\mathrm{sgn}(\phi''(c))\pi/4}, \quad \lambda\to\infty.
\end{aligned}$$

If ϕ has vanished derivatives to a higher order at the stationary point, then the leading term needs to deal with integrals like

$$\int_{-\infty}^{\infty} e^{iz^n}dz,$$

which can be dealt with by the Residue theorem. To obtain more terms of the expansion, we still need to use the steepest descent method combined with Watson's lemma.

Example 8.2.19 (Airy Function)**.** *Consider the Airy function*

$$\mathrm{Ai}(-x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(z^3/3-zx)}dz, \quad x>0.$$

Let $z = x^{1/2}t$*, then* $\mathrm{Ai}(-x) = \frac{x^{1/2}}{2\pi}f(x^{3/2})$*, where*

$$f(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda(t^3/3-t)}dt$$

Since $h(t) = t^3/3 - t$ *has two stationary points at* $z=\pm 1$ *and* $f''(\pm 1) = \pm 2$*, we adopt the previous derivation to find*

$$f(\lambda) \sim \sqrt{\frac{\pi}{\lambda}}e^{-i\frac{2}{3}\lambda}e^{i\frac{\pi}{4}} + \sqrt{\frac{\pi}{\lambda}}e^{i\frac{2}{3}\lambda}e^{-i\frac{\pi}{4}} = 2\sqrt{\frac{\pi}{\lambda}}\cos(\frac{2}{3}\lambda-\frac{\pi}{4}).$$

Therefore,

$$\mathrm{Ai}(-x) \sim \frac{1}{\sqrt{\pi}x^{1/4}}\cos\left(\frac{2}{3}x^{3/2}-\frac{\pi}{4}\right).$$

It implies that the airy function behaves differently for positive and negative arguments, see Figure 8.3. The error is at order $\mathcal{O}(x^{-3/2-1/4}) = x^{-7/4}$*, see Figure 8.4.*

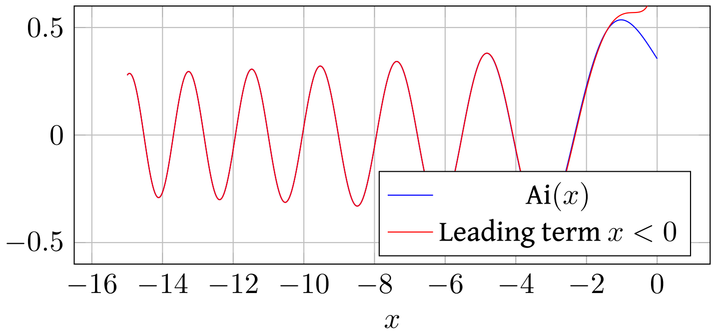

Figure 8.3: Airy function and its asymptotic approximation for $x < 0$.

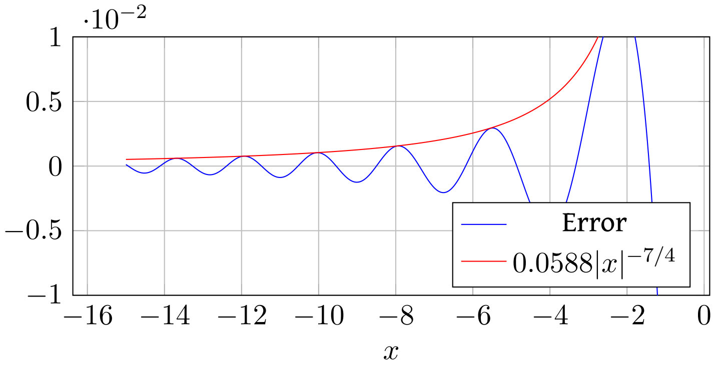

Figure 8.4: Error of the asymptotic approximation is at order $\mathcal{O}(|x|^{-7/4})$, the constant is from $5/(48\sqrt{\pi})$.

Remark 8.2.20. *We point out that the airy function* $\mathrm{Ai}(z)$ *is holomorphic on* $\mathbb{C}$*. But it has quite different behaviors for positive and negative arguments (sectors, to be precise). This is commonly known as the* ***Stokes phenomenon****. This occurs because the function consists of several parts, which dominate in their regions divided by the so-called* ***Stokes lines****, see Olde Daalhuis et al. (1995).*

Example 8.2.21 (Bessel Function)**.** *The Bessel function* J_0 *is defined by*

$$J_0(x) = \frac{2}{\pi}\int_0^{\pi/2} \exp(ix\cos\theta)d\theta = \frac{2}{\pi}\Re\int_0^{\pi/2} e^{ix\cos\theta}d\theta.$$

The function $\cos\theta$ *has a stationary point at* $\theta = 0$ *(which is a boundary point), thus*

$$\begin{aligned} J_0(x) &\sim \frac{2}{\pi}\Re\int_0^{\pi/2} e^{ix(1-\frac{\theta^2}{2})}d\theta \\ &\sim \frac{2}{\pi}\Re e^{ix}\int_0^{\pi/2} e^{-ix\theta^2/2}d\theta \\ &\sim \frac{1}{2}\cdot\frac{2}{\pi}\Re e^{ix}\sqrt{\frac{2\pi}{x}}e^{-i\frac{\pi}{4}} = \sqrt{\frac{2}{\pi x}}\cos(x-\pi/4). \end{aligned}$$

We multiply the factor $\frac{1}{2}$ *since the integration only involves one side of the stationary point. The plot of the asymptotic approximation is in Figure 8.5.*

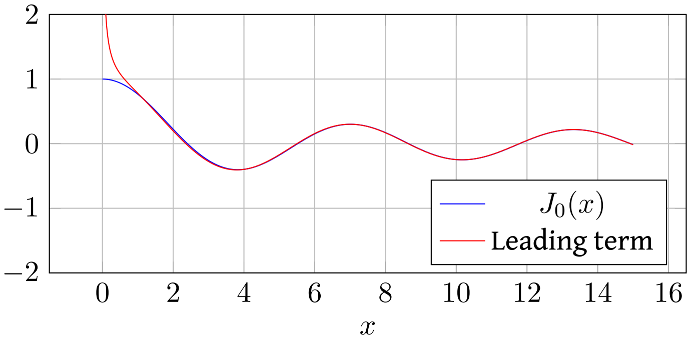

Figure 8.5: Bessel function J_0 and its asymptotic approximation.

8.3 Matched Asymptotic Expansion

In this section, we attempt to cover the basics of **matched asymptotic expansion.**

We start with the following motivating quadratic equation (or general algebraic equations)

$$x^2 + \varepsilon x - 1 = 0,$$

where $0 < \varepsilon \ll 1$. The analytic solutions can be found explicitly.

```
>> Series[-x/2 + Sqrt[1 + x^2/4], {x, 0, 10}]
```

which produces

$$x^* = -\frac{1}{2}\varepsilon \pm \sqrt{1 + \varepsilon^2/4} = \begin{cases} +1 - \frac{1}{2}\varepsilon + \frac{1}{8}\varepsilon^2 - \frac{1}{128}\varepsilon^4 + \mathcal{O}(\varepsilon^6), \\ -1 - \frac{1}{2}\varepsilon - \frac{1}{8}\varepsilon^2 + \frac{1}{128}\varepsilon^4 + \mathcal{O}(\varepsilon^6). \end{cases}$$

The expansion provides a good approximation to the root(s) when $0 \le \varepsilon \ll 1$; sometimes the expansion can be more efficient than calling the square root subroutine (usually `sqrt` is regarded as a single flop, but should be around 4 - 6 flops). However, an analytic solution is rarely found for complex problems; thus, we need to find a systematic way to represent the solution.

The idea of expansion is to *presume* the solution has the **ansatz**:

$$x(\varepsilon) = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots$$

where the coefficients $x_0, x_1, \cdots$ are *a priori* unknown. Then we put the ansatz into the equation.

$$(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots)^2 + \varepsilon(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots) - 1 = 0.$$

And re-group the orders,

$$\begin{aligned} \varepsilon^0 \quad &: \quad x_0^2 - 1 = 0 \Rightarrow x_0 = \pm 1 \\ \varepsilon^1 \quad &: \quad 2x_0 x_1 + x_0 = 0 \Rightarrow x_1 = -\frac{1}{2} \\ \varepsilon^2 \quad &: \quad 2x_0 x_2 + x_1^2 + x_1 = 0 \Rightarrow x_2 = \pm\frac{1}{8} \\ \varepsilon^3 \quad &: \quad 2x_0 x_3 + 2x_1 x_2 + x_2 = 0 \Rightarrow x_3 = 0 \end{aligned}$$

and so on. This expansion is a good approximation for small ε (since the series converges for this problem). Another way to find the expansion is to use an iterative method.

The equation is equivalent to $x = \pm\sqrt{1 - \varepsilon x}$. It generates an iterative scheme

$$x_{n+1} = \pm\sqrt{1 - \varepsilon x_n}, \quad x_0 = 1.$$

Since the derivative $\left|\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{1 - \varepsilon x}\right| = \left|\frac{\varepsilon}{2\sqrt{1-\varepsilon x}}\right| \approx \frac{\varepsilon}{2} \ll 1$ once $\varepsilon \ll 1$, the fixed point iteration converges fast, that is, roughly each iteration it recovers a successive order of ε. We can find that

$$x_1 = \sqrt{1-\varepsilon} = 1 - \frac{1}{2}\varepsilon - \frac{1}{8}\varepsilon^2 - \cdots \approx 1 - \frac{1}{2}\varepsilon$$
$$x_2 = \sqrt{1-\varepsilon(1-\frac{1}{2}\varepsilon)} = 1 - \frac{1}{2}(1-\frac{1}{2}\varepsilon) - \frac{1}{8}\varepsilon^2(1-\frac{1}{2}\varepsilon)^2 + \cdots \approx 1 - \frac{1}{2}\varepsilon + \frac{1}{8}\varepsilon^2$$
$$x_3 = \sqrt{1-\varepsilon(1-\frac{1}{2}\varepsilon+\frac{1}{8}\varepsilon^2)} = \cdots$$

However, sometimes the power series ansatz may not be effective.

Example 8.3.1. *Consider the algebraic equation*

$$x^3 - x^2 - (1+\varepsilon)x + 1 = 0.$$

When $\varepsilon = 0$, $x^* = 1$ is a root. If we take the ansatz

$$x(\varepsilon) = 1 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots,$$

then we will obtain

$$\varepsilon^1 : \quad 3x_1 - 2x_1 - x_1 - 1 = 0.$$

Contradiction! The power series ansatz is not a suitable expansion. The reason is that when $\varepsilon = 0$, $x^* = 1$ is a double root, it means the equation behaves quadratically at $x = 1$, a linear perturbation of order ε will create $\mathcal{O}(\varepsilon^2)$ in the equation. To recover the $\mathcal{O}(\varepsilon)$ perturbation in the equation, we need to introduce $\mathcal{O}(\varepsilon^{1/2})$ perturbation in $x(\varepsilon)$:

$$x(\varepsilon) = 1 + \varepsilon^{1/2} x_{1/2} + \varepsilon x_1 + \cdots.$$

Then $x_{1/2} = \pm\frac{1}{\sqrt{2}}$ (it means the double root will split in two opposite directions), and if we take the positive case, it leads to $x_1 = \frac{1}{8}$, and so on (the following code extracts all coefficients).

```
>> t = Solve[x^3 - x^2 - (1 + eps)*x + 1 == 0, {x}][[1]];
>> f[eps_] := x /. t[[1]];
>> k = Series[f[eps], {eps, 0, 10}, Assumptions -> eps > 0];
>> For[i = 0, i < 10, i++, Print[Simplify[SeriesCoefficient[k, i/2]]]]
```

Similarly, the equation

$$(1-\varepsilon)x^2 - 2x + 1 = 0$$

should also consider the terms of size $\mathcal{O}(\varepsilon^{1/2})$ in the ansatz. In general, if the equation is perturbed linearly in ε, the root needs to include terms like $\mathcal{O}(\varepsilon^{1/n})$ if the root has multiplicity of n.

In the previous cases, the behavior of the perturbed root is more or less *continuous*, that is, small ε leads to a small perturbation in the root. Let us consider the equation

$$\varepsilon x^2 + x - 1 = 0,$$

which has one root at $\varepsilon = 0$; however, it has two distinct roots (not even close) for every $\varepsilon > 0$. Such behavior is not continuous, and we call this perturbation **singular**.

For this example, we can use its analytic form for some insight

$$x^* = \frac{1}{2\varepsilon}(-1 \pm \sqrt{1+4\varepsilon}) = \begin{cases} 1 - \varepsilon + 2\varepsilon^2 - 5\varepsilon^3 + \cdots , \\ -\varepsilon^{-1} - 1 + \varepsilon - 2\varepsilon^2 + 5\varepsilon^3 + \cdots . \end{cases}$$

One of the roots is staying near the original root (can be found by the usual ansatz), the other one is approaching ∞ (singular). To determine the singular root, we need to include the singular part

$$x(\varepsilon) = \varepsilon^{-1}x_{-1} + x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots .$$

The standard and clean method is to apply *rescaling* $x = X/\delta$, where δ is some small scaling factor to be determined. Then

$$\varepsilon X^2/\delta^2 + X/\delta - 1 = 0 \Rightarrow X^2 + \varepsilon^{-1}\delta X = \varepsilon^{-1}\delta^2$$

Here we take $\delta = \varepsilon$, which will make the problem back to a regular perturbation.

8.3.1 Regular Perturbation (ODE)

The previous idea can extend to differential equations, where functions replace the coefficients.

Example 8.3.2. *Consider the ODE model of a slow projectile based on Newton's 2nd law.*

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = -\frac{gR^2}{(y+R)^2}, \quad y(0) = 0, \quad y'(0) = v_0.$$

where R is the radius of the Earth and $y \ll R$ is the height, v_0 is the initial velocity, see Figure 8.6.

This model can be reduced to the following simplified ODE by a rescaling

$$y'' = -\frac{1}{(1+\varepsilon y)^2}, \quad y(0) = 0, \quad y'(0) = 1.$$

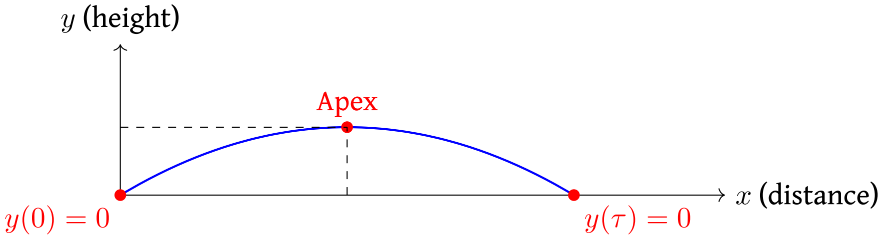

Figure 8.6: Illustration of the trajectory.

Assume the ansatz $y(\varepsilon, t) = y_0(t) + \varepsilon^\alpha y_1(t) + \cdots$*, where* $\alpha > 0$ *is undetermined for now. Then*

$$y_0''(t) + \varepsilon^\alpha y_1''(t) + \cdots = -\frac{1}{[1 + \varepsilon(y_0(t) + \varepsilon^\alpha y_1(t) + \cdots)]^2} = -1 + 2\varepsilon y_0(t) + \cdots$$

The right-hand side has $\mathcal{O}(\varepsilon)$ *term, thus we need to counter that from the left-hand side by setting* $\alpha = 1$*. Therefore,*

$$y_0''(t) = -1, \quad y_0(0) = 0, \quad y_0'(0) = 1.$$

That is, $y_0(t) = -t^2/2 + t$*. Similarly, we can derive the equation for* y_1*:*

$$y_1''(t) = -2y_0(t), \quad y_1(0) = 0, \quad y_1'(0) = 0.$$

It solves $y_1(t) = \frac{1}{3}t^3 - \frac{1}{12}t^4$*. Thus, we obtain a two-term asymptotic approximation*

$$y(t) \sim -\frac{t^2}{2} + t + \varepsilon\left(\frac{1}{3}t^3 - \frac{1}{12}t^4\right).$$

This asymptotic form can be used to estimate the return time $y(\tau) = 0$ *by solving the perturbed algebraic equation.*

Example 8.3.3. *Consider the spring system with a small friction proportional to the velocity (in the opposite direction)*

$$y'' = -\varepsilon y' - y, \quad y(0) = 0, \quad y'(0) = 1.$$

We take $y(\varepsilon, t) = y_0(t) + \varepsilon^\alpha y_1(t) + \cdots$ *and*

$$y_0''(t) + y = 0, \quad y_0(0) = 0, \quad y_0'(0) = 1.$$

It solves $y_0(t) = \sin(t)$*. The equation for* y_1 *is derived as*

$$\varepsilon^\alpha y_1''(t) + \cdots = -\varepsilon y_0'(t) - \varepsilon y_1 + \cdots$$

which means $\alpha = 1$ *and* $y_1''(t) = -y_1 - y_0(t), y_1(0) = 0, y_1'(0) = 0$*. Thus,*

$$y_1(t) = -\frac{t}{2}\sin(t).$$

Therefore, the two-term asymptotic form is $y(t) = \sin(t)\frac{t}{2}\varepsilon\sin(t)$*. We can verify with the analytic solution using* `Mathematica`*.*

$$y_{analytic}(t) = \frac{2}{\sqrt{4-\varepsilon^2}}e^{-\varepsilon t/2}\sin\left(\frac{\sqrt{4-\varepsilon^2}}{2}t\right).$$

```
>> f[s_, t_] := 2/Sqrt[4 - s^2] * Exp[-s * t/2] *Sin[Sqrt[4 - s^2] * t/2]
>> Series[f[s, t], {s, 0, 10}]
```

8.3.2 Singular Perturbation (ODE)

When a small perturbation is applied to the highest order of linear differential equations, it will cause singular behavior due to an additional basis that concentrates near the boundary of the domain.

Example 8.3.4. *Consider the boundary value problem*

$$\varepsilon y'' + (1+\varepsilon)y' + y = 0, \quad y(0) = 0, \quad y(1) = 1.$$

The analytic solution is

$$y = \frac{e^{-t} - e^{-t/\varepsilon}}{e^{-1} - e^{-1/\varepsilon}}.$$

Near the boundary $t \to 0$*, we obtain that*

$$y(t) \sim 1 - e^{-t/\varepsilon}.$$

This function changes rapidly near $t = 0$ *as* $\varepsilon \to \infty$*; this phenomenon is called the* ***boundary layer****. The core idea of matched asymptotic expansion is to decompose the solution into the boundary layer solution and the outer solution, which will match each other through an intermediate scale.*

Outer Solution

The outer solution ignores the boundary layer as if it does not exist. In the previous example (since we know where the boundary layer is), this means,

$$\varepsilon y''_{\text{out}} + (1+\varepsilon) y'_{\text{out}} + y_{\text{out}} = 0, \quad y(1) = 1.$$

Then we expand

$$y_{\text{out}}(t) = y_{\text{out},0} + \varepsilon y_{\text{out},1}(t) + \varepsilon^2 y_{\text{out},2}(t) + \cdots,$$

which satisfies the relations:

$$\begin{aligned}
\varepsilon^0 \quad &: \quad y'_{\text{out},0} + y_{\text{out},0} = 0, \quad y_{\text{out},0}(1) = 1. \\
\varepsilon^1 \quad &: \quad y''_{\text{out},0} + y'_{\text{out},0} + y'_{\text{out},1} + y_{\text{out},1} = 0, \quad y_{\text{out},1}(1) = 0. \\
&\vdots \\
\varepsilon^n \quad &: \quad y''_{\text{out},n-1} + y'_{\text{out},n-1} + y'_{\text{out},n} + y_{\text{out},n} = 0, \quad y_{\text{out},n}(1) = 0.
\end{aligned}$$

More precisely,

$$y_{\text{out},0} = e^{1-t} \quad \text{and} \quad y_{\text{out},n} = 0, \quad n > 0.$$

In general, the outer solution will have some undetermined constants, but in the above example, the outer solution is completely determined.

Boundary Layer Solution

Now, we only have to deal with the solution in the boundary layer. It is also called *inner solution*, meaning inside the boundary layer.

The standard approach takes a stretched coordinate transform. In our previous example, the stretched variable is $\zeta = t/\delta$, where δ is a scaling factor. Then the equation for $y(t) = Y(\zeta)$ becomes

$$\frac{\varepsilon}{\delta^2} Y_{\zeta\zeta} + \frac{(1+\varepsilon)}{\delta} Y_{\zeta} + Y = 0.$$

The coefficients $\frac{\varepsilon}{\delta^2}$, $\frac{1}{\delta}$, $\frac{\varepsilon}{\delta}$ and 1 should balance in the sense that Y and its derivatives are at $\mathcal{O}(1)$.

Since $\frac{\varepsilon}{\delta} \ll \frac{1}{\delta}$, the dominating orders could be balanced from one of the following three possibilities:

i. $\frac{\varepsilon}{\delta^2}$ and $\frac{1}{\delta}$: it means $\delta = \mathcal{O}(\varepsilon)$. The other two terms are of order 1.

ii. $\frac{1}{\delta}$ and 1: it leads to $\delta = \mathcal{O}(1)$. This does not do any scaling; the resulting equations will produce an outer solution instead.

iii. $\frac{\varepsilon}{\delta^2}$ and 1: it means $\delta = \mathcal{O}(\sqrt{\varepsilon})$. Then the other two terms are $\mathcal{O}(1)$ and $\sqrt{\varepsilon}$, which cannot be cancelled (unless the coefficient of Y becomes zero).

The only possibility is $\delta = \mathcal{O}(\varepsilon)$, we simply let $\delta = \varepsilon$. Consider the ansatz

$$Y(\zeta) = g_0(\zeta) + \varepsilon g_1(\zeta) + \varepsilon^2 g_2(\zeta) + \cdots$$

It satisfies

$$\begin{array}{lll}
\varepsilon^{-1} & : & g_0'' + g_0' = 0, \quad g_0(0) = 0 \\
\varepsilon^{0} & : & g_1'' + g_1' + g_0' + g_0 = 0, \quad g_1(0) = 0 \\
\varepsilon^{1} & : & g_2'' + g_2' + g_1' + g_1 = 0, \quad g_2(0) = 0 \\
\vdots & &
\end{array}$$

It solves $g_0(\zeta) = C_0(e^{-\zeta} - 1)$ and $g_k(\zeta) = C_k(e^{-\zeta} - 1)$ for $k > 0$. The constants are undetermined.

Matching and Combining

We have found the outer solution $y_{\text{out}}(t)$ and the boundary layer solution $y_{\text{inn}} \sim g_0(t/\varepsilon)$. To determine the constant C_0 in g_0, we assume that both approximations can be extended a little bit to an overlap region $t = \xi\varepsilon^\beta$ for some $\beta \in (0, 1)$ and $\xi \sim \mathcal{O}(1)$ and the dominating behavior of both match, that is, $y_{\text{out}}(\xi\varepsilon^\beta) \approx y_{\text{inn}}(\xi\varepsilon^{\beta-1})$.

$$e^{1-\xi\varepsilon^\beta} \approx C_0(\xi e^{-\varepsilon^{\beta-1}} - 1)$$

When $\varepsilon \to 0$, we find that $C_0 = -e^1$. Thus $y_{\text{inn}} \sim e^1 - e^{1-t/\varepsilon}$.

Then we combine the solutions and subtract the common (matched) part,

$$y(t) \sim y_{\text{inn}}(t) + y_{\text{out}}(t) - y_{\text{out}}(0) = e^{1-t} - e^{1-t/\varepsilon}.$$

More Terms

It is possible to repeat the above process to obtain more terms. In this example, we do not have higher-order terms since the remainder is at order $\mathcal{O}(e^{-t/\varepsilon})$, which is transcendentally small. In general, if we have

$$y_{\text{out}}(t) = y_{\text{out},0}(t) + \varepsilon y_{\text{out},1}(t) + \cdots$$

and

$$y_{\text{inn}}(t) = g_0(t/\varepsilon; C_0) + \varepsilon g_1(t/\varepsilon; C_1) + \cdots$$

It is possible to determine the constants $C_0, C_1, \cdots$ in the same way by choosing an intermediate parameter $t = \xi\varepsilon^\beta$, and match

$$y_{\text{out},0}(\xi\varepsilon^\beta) + \varepsilon y_{\text{out},1}(\xi\varepsilon^\beta) + \cdots \sim g_0(\xi\varepsilon^{\beta-1}; C_0) + \varepsilon g_1(\xi\varepsilon^{\beta-1}; C_1) + \cdots$$

by expanding into the Taylor series. The common terms will be a truncation of the resulting (matched) Taylor series.

Remark 8.3.5. *Usually, if there is an unmatched term in the outer solution, it might be matched later from a successive order from the boundary layer solution. Therefore, it is advisable to expand a few more terms of y_{out} and y_{inn} to ensure their series match. For instance (see Holmes (2012)), in the ODE*

$$\varepsilon y'' + 2y' + 2y = 0, \quad y(0) = 0, \quad y(1) = 1,$$

suppose we have the outer solution

$$y_{out}(t) \sim e^{1-t} + \varepsilon\frac{1}{2}(1-t)e^{1-t} + \cdots$$

and the inner solution

$$y_{inn}(t) \sim e^1 - e^{1-2t/\varepsilon} + \varepsilon\left[C(1 - e^{-2t/\varepsilon}) - \frac{t}{\varepsilon}e^1(1 + e^{-2t/\varepsilon})\right] + \cdots$$

Setting $t = \xi\varepsilon^\beta$, $\beta \in (0,1)$, will expand them into (we choose $\beta = 0.99$ for simipicity)

$$y_{out} \sim e^1 - \xi\varepsilon^\beta e^1 + \frac{1}{2}\varepsilon e^1 + \cdots$$

and

$$y_{inn} \sim e^1 + C\varepsilon - \xi\varepsilon^\beta e^1 + \cdots$$

which makes $C = \frac{1}{2}e^1$. The common part of both solutions is $e^1(1 - t + \frac{1}{2}\varepsilon)$. Notice that the last term of y_{inn} comes from the $\mathcal{O}(\varepsilon)$ term of the expansion, while in y_{out}, it is from the $\mathcal{O}(1)$ term. Having $t = \mathcal{O}(\varepsilon^{1^-})$ can ensure the truncated terms are all higher-order terms. In the $\mathcal{O}(\varepsilon^2)$ term of y_{inn}, we will expect something like

$$\varepsilon^2\left(\frac{t}{\varepsilon}\right)^2(A + Be^{-2t/\varepsilon}) \sim A\xi^2\varepsilon^{2\beta}$$

while this kind of term appears in the $\mathcal{O}(1)$ and $\mathcal{O}(\varepsilon)$ terms of outer solution.

For $\mathcal{O}(1)$ asymptotic approximation, we can extend to a slightly more general case.

Theorem 8.3.6. *Consider the boundary value problem*

$$\varepsilon y'' + p(x)y' + q(x)y = 0, \quad y(0) = a, \quad y(1) = b.$$

where $p(x) > 0$*. The inner solution*

$$y_{\mathit{inn}}(x) = C + (a - C)e^{-p(0)/\varepsilon},$$
$$y_{\mathit{out}}(x) = b \exp\left(\int_x^1 \frac{q(s)}{p(s)} ds\right).$$

The constant $C = y_{\mathit{out}}(0)$*. The combined solution is* $y(x) \sim y_{\mathit{inn}} + y_{\mathit{out}} - C$*.*

Proof Sketch. The essential assumption is that the boundary layer only appears at $x = 0$. Once we know that, the derivation is the same as the previous example. The derivation of the $\mathcal{O}(\varepsilon)$ term is left as an exercise. ☐

Interior Layers

Sometimes, the layer solution occurs in the interior. For instance, the steady state of the Burgers' equation with anti-damping (negative) or damping (positive):

$$\varepsilon y'' = yy' \mp y, \quad y(0) = 1, \quad y(1) = -1.$$

The numerical solution (and its asymptotic approximation) is shown in Figure 8.7. The sharp change is not at the boundary points; instead, it is located in the interior

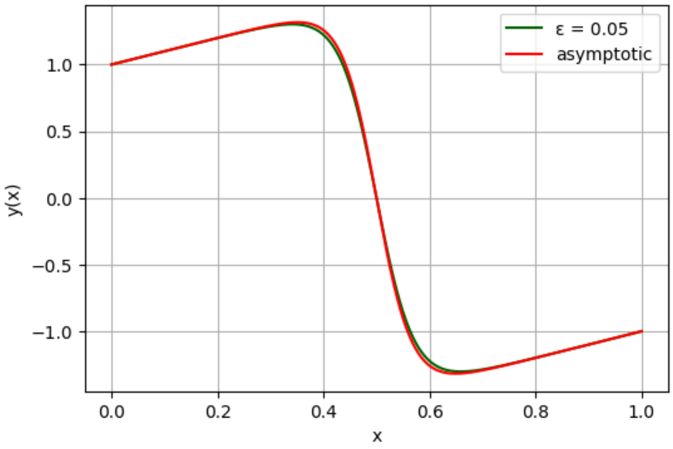

Figure 8.7: Numerical solution (anti-damping) and the asymptotic solution at $\varepsilon = 0.05$.

of the domain. The solution transitions from one outer solution to another. We take the anti-damping case for example.

Suppose the interior layer occurs at $x_0 \in (0,1)$. We need to calculate outer solutions from the left and right sides.

i. Left side. We set $y_{l,\text{out}} = \sum_{n=0}^{\infty} \varepsilon^n y_{l,\text{out},n}(x)$, then

$$\begin{aligned}\varepsilon^0 &: \quad y_{l,\text{out},0}(x)y'_{l,\text{out},0}(x) - y_{l,\text{out},0}(x) = 0, \quad y_{l,\text{out},0}(0) = 1\\ \varepsilon^1 &: \quad y''_{l,\text{out},0} = y_{l,\text{out},0}y'_{l,\text{out},1} + y_{l,\text{out},1}y'_{l,\text{out},0} - y_{l,\text{out},1}, \quad y_{l,\text{out},1}(0) = 0.\end{aligned}$$

It solves $y_{l,\text{out},0}(x) = x+1$, and $y_{l,\text{out},1} = 0$, which inductively proves all $y_{l,\text{out},k} = 0$ for $k \geq 1$.

ii. Right side. Similarly $y_{r,\text{out}} = x - 2$.

Then we deal with the inner solution (interior layer). On the right-side of x_0, we introduce the scaling $\zeta = \frac{x-x_0}{\delta}$, then let $Y(\zeta) = y(x)$,

$$\frac{\varepsilon}{\delta^2}Y'' = \frac{1}{\delta}YY' - Y$$

The dominating terms should be $\frac{\varepsilon}{\delta^2} \sim \frac{1}{\delta} \gg 1$, we take $\delta = \varepsilon$. The inner solution solves

$$\frac{1}{\varepsilon}Y'' = \frac{1}{\varepsilon}YY' - Y.$$

Let $Y(\zeta) = Y_0(\zeta) + \varepsilon Y_1(\zeta) + \cdots$, then

$$Y_0'' = Y_0Y_0', \quad Y_1'' = Y_1Y_0' + Y_1'Y_0 - Y_0.$$

There are three cases for Y_0.

- $Y_0(\zeta) = A\frac{1-Be^{A\zeta}}{1+Be^{A\zeta}}$, where A, B are arbitrary.
- $Y_0(\zeta) = A\tan(B + A\zeta/2)$.
- $Y_0(\zeta) = \frac{2}{A-\zeta}$.

When we perform the matching, we will need to compute $Y_0(\infty)$; this leaves us only the first option. Matching the right side by taking $x = x_0 + \xi\varepsilon^\beta$,

$$Y_0(\varepsilon^{\beta-1}) \sim x_0 + \xi\varepsilon^\beta - 2$$

The limit $\varepsilon \to 0$ leads to $-A = x_0 - 2$. Similarly, we derive the left side's matching, $A = x_0 + 1$. Therefore, we must have $x_0 = \frac{1}{2}$ and $A = \frac{3}{2}$. It remains to determine

B. This can be derived from the symmetry of the solution, that is

$$w(x) := -y(1-x)$$

satisfies the same equation, thus $-y(1-x) = y(x)$, and $y(\frac{1}{2}) = 0$, it means $B = 1$.

Finally, we combine the inner solution and outer solutions. Due to the symmetry, we only have to deal with one side. For $x > \frac{1}{2}$,

$$y(x) = x - 2 + \frac{3}{2}\frac{1 - e^{3(x-1/2)/2\varepsilon}}{1 + e^{3(x-1/2)/2\varepsilon}} + \frac{3}{2} = x + 1 - \frac{3e^{3(x-1/2)/2\varepsilon}}{1 + e^{3(x-1/2)/2\varepsilon}}.$$

It can be seen that when $\varepsilon \to 0$, we will see a sharp jump from $\frac{3}{2}$ to $-\frac{3}{2}$ at $x = \frac{1}{2}$.

Remark 8.3.7. *The nonlinear term yy' in the equation caused the interior layer. Suppose we treat y as the coefficient of y'. When y is always positive or negative, the inner solution will be an exponentially growing/decaying function, and it cannot be an interior layer since it will blow up on one side. Therefore, the layer will be a boundary layer. Intuitively, having a positive y pushes the boundary layer to the right, and a negative y pushes back. When y changes from positive to negative, the effect is a layer pushed in the middle. The point that $y = 0$, in this case, is called the* ***turning point****. We can expect an interior layer when y is replaced by y^3, but only a boundary layer at the right endpoint if replaced by y^2.*

Remark 8.3.8. *It is possible to have multiple boundary/interior layers, and it is essential to locate them before performing asymptotic analysis. However, given a general problem, the most effective way is probably first to let $\varepsilon = 10^{-3}$ or so to simulate a numerical solution to gain insight into whether a layer solution exists.*

Corner Layers

Sometimes, the solution itself does not have abrupt changes, but its derivatives do. For instance,

$$\varepsilon y'' + (x - \frac{1}{2})y' - y = 0, \quad y(0) = 2, \quad y(1) = 3.$$

The point $x_0 = \frac{1}{2}$ is a turning point, which may cause a layer in the interior instead of at the boundaries. First, we solve the leading term of the outer solutions using two boundary conditions by dropping $\varepsilon y''$.

$$y_{\text{out}} = \begin{cases} -4(x - \frac{1}{2}), & x < x_0, \\ +6(x - \frac{1}{2}), & x > x_0. \end{cases}$$

These two lines join at $x_0 = \frac{1}{2}$, forming a continuous but non-differentiable function (thus called a **corner layer**), see Figure 8.8.

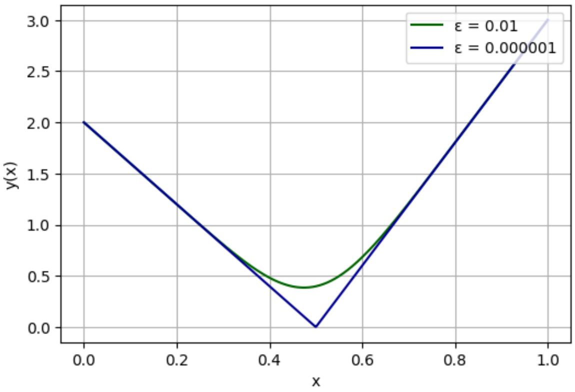

Figure 8.8: Numerical solutions for different ε.

For inner solution (on either side), take $Y(\zeta) = y(\frac{x-x_0}{\delta})$,

$$\frac{\varepsilon}{\delta^2}Y'' + \zeta Y' - Y = 0$$

Thus the only viable choice is $\delta = \sqrt{\varepsilon}$ and the resulting inner solution's equation is

$$Y'' + \zeta Y' - Y = 0.$$

It does not involve ε, but it does not mean we can treat its solution as the leading term directly. It could be higher-order terms. It can be figured out later in the matching process. The resulting solution is (by reduction of order)

$$Y(\zeta) = A\zeta + B\left[-e^{-\zeta^2/2} - \zeta\int_0^\zeta e^{-s^2/2}ds\right].$$

The matching near $x = x_0 + \xi\varepsilon^\beta$, where $\beta \in (0, {}^1\!/\!_2)$,

$$y_{\text{out}}(x) \sim \begin{cases} -4\varepsilon^\beta\xi = -4(x-x_0), & \xi < 0 \\ +6\varepsilon^\beta\xi = +6(x-x_0), & \xi > 0 \end{cases}$$

and $y_{\text{inn}} \sim \xi\varepsilon^{\beta-\frac{1}{2}}(A - B\sqrt{\pi/2})$ for positive ξ and $y_{\text{inn}} \sim \xi\varepsilon^{\beta-\frac{1}{2}}(A + B\sqrt{\pi/2})$ for negative ξ. Comparing the solutions, we find

$$A - B\sqrt{\pi/2} = -4\varepsilon^{1/2}, \quad A + B\sqrt{\pi/2} = 6\varepsilon^{1/2}.$$

Therefore,

$$Y(\zeta) = \varepsilon^{1/2}\zeta - 5\sqrt{2/\pi}\varepsilon^{1/2}\left[-e^{-\zeta^2/2} - \zeta\int_0^\zeta e^{-s^2/2}ds\right].$$

Then, combining the inner and outer solutions, we obtain (also see Figure 8.9)

$$y(x) \sim (x - \frac{1}{2}) + 5\sqrt{2/\pi}\left[\varepsilon^{1/2}e^{-(x-\frac{1}{2})^2/(2\varepsilon)} + (x - \frac{1}{2})\int_0^{(x-\frac{1}{2})\varepsilon^{-1/2}} e^{-s^2/2}ds\right].$$

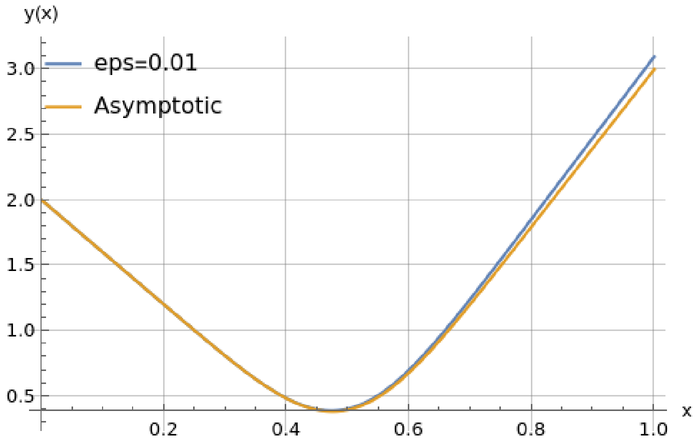

Figure 8.9: Numerical solution and asymptotic approximation at $\varepsilon = 0.01$.

Initial Layers

The initial layers refer to an abrupt change on a short time scale near $t = 0$. This is common for dynamics that involve small scales.

8.4 Multiscale Expansion

The multiscale expansion method merges the outer and inner solutions and assumes the solution in the form $y(x) = y(x, \frac{x}{\varepsilon^\alpha})$, where $\frac{x}{\varepsilon^\alpha}$ is called **fast variable**. Asymptotically, the solution's expansion is

$$y(x) = y_0(x, \frac{x}{\varepsilon^\alpha}) + \varepsilon^\beta y_1(x, \frac{x}{\varepsilon^\alpha}) + \cdots$$

8.5 WKB Methods

TODO

8.6 Homogenization

TODO

8.7 Exercises

☕ **Problem 8.7.1.** *Find the asymptotic approximation of Legendre polynomials*

$$P_n(z) = \frac{1}{2^{n+1}\pi i}\int_C \frac{(t^2-1)^n}{(t-z)^{n+1}}dt$$

where C is a contour enclosing $z \in \mathbb{R}$ with $n \to \infty$.

☞ **Problem 8.7.2.** *Find the first three terms in the asymptotic solution ansatz $y = y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \cdots$ for*

$$y'' + \varepsilon y' + \varepsilon^2 y = 0, \quad y(0) = 0, \quad y(1) = 1.$$

☕ **Problem 8.7.3.** *Prove Theorem 8.3.6 and derive the asymptotic expansion to order $\mathcal{O}(\varepsilon)$.*

☕ **Problem 8.7.4.** *Let $0 < \varepsilon \ll 1$. Use **matched asymptotic expansion** to find the asymptotic solution for*

$$\varepsilon^2 y'' + 2\varepsilon p(x) y' - q(x) y = f(x), \quad y(0) = a, \quad y(1) = b.$$

The functions p, q, f are continuous, and q is positive.

☕ **Problem 8.7.5.** *Let $0 < \varepsilon \ll 1$. Use **matched asymptotic expansion** to find the asymptotic solution for*

$$\varepsilon y'' + \left(\frac{1}{2} - t\right) y' + y = 0, \quad y(0) = 0, \quad y(1) = 1.$$

☕ **Problem 8.7.6.** *Consider the ODE*

$$y'' + \frac{1}{\varepsilon} y' + y(1-y) = 0, \quad y(0) = 0, \quad y(1) = 1.$$

*use **matched asymptotic expansion** to find the asymptotic solution when $\varepsilon \to 0^+$. You may assume there exists a boundary layer at $x = 0$.*

☕ **Problem 8.7.7.** *Let $0 < \varepsilon \ll 1$. Use **matched asymptotic expansion** to find the asymptotic solution for*

$$\varepsilon y'' + \varepsilon (x+1)^2 y' - y = x - 1, \quad y(0) = 0, \quad y(1) = -1.$$

Extended Reading

Hinch, E. J. (1991). *Perturbation Methods*. Cambridge Texts in Applied Mathematics. Cambridge University Press.

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Olde Daalhuis, A., Chapman, S. J., King, J. R., Ockendon, J. R., and Tew, R. H. (1995). Stokes phenomenon and matched asymptotic expansions. *SIAM Journal on Applied Mathematics*, 55(6):1469–1483.

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