

Chapter 7

Transforms and Special Functions

A mathematician who is not also something of a poet will never be a complete mathematician.

– Karl Weierstrass



TRANSFORMS, such as the Fourier, Laplace, and Mellin transforms, are mathematical techniques that convert problems from one domain to its dual, often simplifying analysis and computation. Special functions, including Bessel functions, Legendre polynomials, and Gamma functions, arise naturally as solutions to differential equations encountered in physical systems. This chapter serves as an introduction to both topics.

7.1 Fourier Transform

The Fourier transform plays a crucial role in all branches of analysis. This chapter serves as an introduction to the Fourier transform and a canonical extension of complex analysis, as presented in Chapter 0.

Definition 7.1.1. *The one-dimensional Fourier transform is defined by the following*

$$\widehat{f}(\zeta) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x \zeta} dx, \quad \zeta \in \mathbb{R}.$$

A similar formula defines the inverse Fourier transform

$$f(x) = \int_{-\infty}^{\infty} \widehat{f}(\zeta)e^{2\pi i x \zeta} d\zeta, \quad x \in \mathbb{R}.$$

Not all functions have Fourier transforms. In the following, we define a special class of functions $\mathfrak{G}_{a,\delta}$ with $a, \delta > 0$ by conditions:

- (i) The function f is holomorphic in the strip $S_a = \{z \in \mathbb{C} \mid |\Im z| < a\}$.
- (ii) There exists a constant $A > 0$ such that

$$|f(x + iy)| \leq \frac{A}{1 + |x|^{1+\delta}}, \quad \forall x \in \mathbb{R}, |y| < a.$$

Of course, the Fourier transform serves a much broader class of functions (e.g., compactly supported continuous functions). We only use $\mathfrak{G}_{a,\delta}$ for some rigor.

Theorem 7.1.2. *If $f \in \mathfrak{G}_{a,\delta}$, then $|\widehat{f}(\zeta)| \leq Ce^{-2\pi b|\zeta|}$ for any $0 \leq b < a$.*

Proof. Without loss of generality, we assume $\zeta > 0$ and consider the following contour integral for $g(z) = f(z)e^{-2\pi i z \zeta}$.

On the vertical sides of the rectangular contour integral, the contribution converges to zero as $R \rightarrow \infty$:

$$\begin{aligned} \left| \int_{-R}^{-R-ib} g(z) dz \right| &= \left| \int_0^b f(-R - iy)e^{-2\pi i(-R-iy)\zeta} dy \right| \\ &\leq \int_0^b \frac{A}{1 + R^{1+\delta}} e^{-2\pi y \zeta} dy \xrightarrow{R \rightarrow \infty} 0 \end{aligned}$$

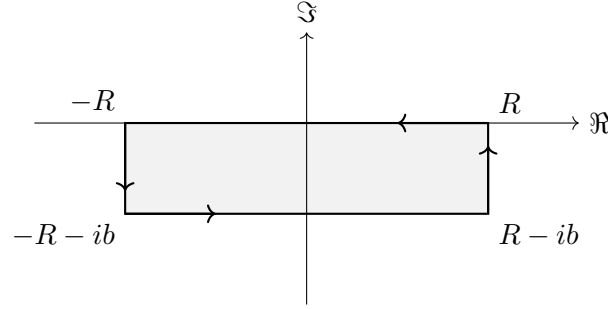


Figure 7.1: Contour integral around a rectangle.

Therefore,

$$\widehat{f}(\zeta) = \int_{-\infty}^{\infty} f(x - ib)e^{-2\pi i(x-ib)\zeta} dx.$$

and leads to the estimate

$$|\widehat{f}(\zeta)| \leq \int_{-\infty}^{\infty} \frac{A}{1 + |x|^{1+\delta}} e^{-2\pi b\zeta} dx \leq C e^{-2\pi b\zeta}.$$

□

The theorem shows that if $f \in \mathfrak{G}_{a,\delta}$, then its Fourier transform decays rapidly on \mathbb{R} . In turn, the inversion formula will then be well-defined.

Theorem 7.1.3. If $f \in \mathfrak{G}_{a,\delta}$, then

$$f(x) = \int_{-\infty}^{\infty} \widehat{f}(\zeta) e^{2\pi i x \zeta} d\zeta, \quad \forall x \in \mathbb{R}.$$

Proof. The choice of contour path depends on the sign of ζ . For $\zeta > 0$, we reuse the contour in Figure 7.1.

$$\begin{aligned} \int_0^{\infty} \widehat{f}(\zeta) e^{2\pi x \zeta} d\zeta &= \int_0^{\infty} \int_{-\infty}^{\infty} f(w - ib) e^{-2\pi i(w-ib)\zeta} e^{2\pi i x \zeta} dw d\zeta \\ &= \int_{-\infty}^{\infty} f(w - ib) \underbrace{\int_0^{\infty} e^{-2\pi i(w-ib-x)\zeta} d\zeta}_{=(2\pi i(w-ib-x))^{-1}} dw \\ &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} f(w - ib) \frac{1}{w - ib - x} dw = \frac{1}{2\pi i} \int_{\mathbb{R}-ib} \frac{f(z)}{z - x} dz \end{aligned}$$

Similarly, we can derive

$$\int_{-\infty}^0 \widehat{f}(\zeta) e^{2\pi x \zeta} d\zeta = -\frac{1}{2\pi i} \int_{\mathbb{R}+ib} \frac{f(z)}{z-x} dz.$$

Again, the contribution on the vertical sides is negligible. Therefore (γ is the path in Figure 7.2),

$$\int_0^{\infty} \widehat{f}(\zeta) e^{2\pi x \zeta} d\zeta + \int_{-\infty}^0 \widehat{f}(\zeta) e^{2\pi x \zeta} d\zeta = \lim_{R \rightarrow \infty} \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z-x} dz = f(x).$$

□

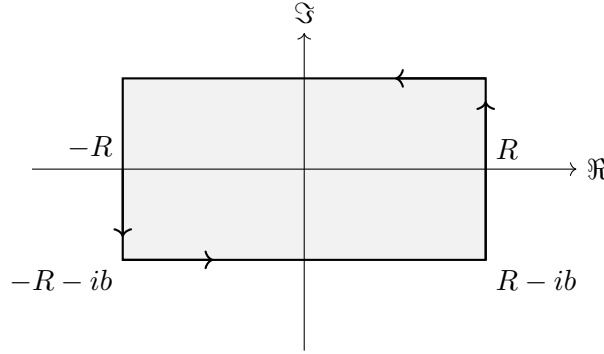


Figure 7.2: Contour for inverse Fourier Transform.

Remark 7.1.4. According to the definition, $\widehat{\widehat{f}}(x) = f(-x)$, that is, applying the Fourier transform twice will reflect the function.

Remark 7.1.5. The multi-dimensional Fourier transform is defined as

$$\widehat{f}(\zeta_1, \zeta_2, \dots, \zeta_n) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} f(x_1, \dots, x_n) e^{-2\pi(\zeta_1 x_1 + \dots + \zeta_n x_n)} dx_1 dx_2 \dots dx_n,$$

which is taking the one-dimensional Fourier transform on each dimension.

We provide a few common examples of Fourier transforms in one dimension and higher dimensions.

Example 7.1.6. Let $f(x) = \frac{1}{x^2 + a^2}$, its Fourier transform is (suppose $\zeta < 0$),

$$\widehat{f}(\zeta) = \int_{-\infty}^{\infty} \frac{e^{-2\pi i x \zeta}}{x^2 + a^2} dx = \lim_{R \rightarrow \infty} \int_{\gamma_R} \frac{e^{-2\pi i z \zeta}}{z^2 + a^2} dz = 2\pi i \operatorname{Res}(f, ia) = \frac{\pi}{a} e^{-2\pi|\zeta|a}.$$

where γ_R is the semi-circle contour in the upper plane.

Example 7.1.7. Let $f(x) = e^{-\pi x^2}$, its Fourier transform is

$$\begin{aligned}\widehat{f}(\zeta) &= \int_{-\infty}^{\infty} e^{-\pi x^2 - 2\pi i \zeta x} dx = e^{-\pi \zeta^2} \int_{-\infty}^{\infty} e^{-\pi(x+i\zeta)^2} dx = e^{-\pi \zeta^2} \int_{-\infty+i\zeta}^{\infty+i\zeta} e^{-\pi z^2} dz \\ &= e^{-\pi \zeta^2} \int_{-\infty}^{\infty} e^{-\pi z^2} dz = e^{-\pi \zeta^2} \widehat{f}(0).\end{aligned}$$

To determine $\widehat{f}(0)$, we use the Plancherel's identity in Theorem 7.1.9, that is

$$\widehat{f}(0)^2 \int_{-\infty}^{\infty} e^{-2\pi \zeta^2} d\zeta = \int_{-\infty}^{\infty} e^{-2\pi x^2} dx$$

Hence $\widehat{f}(0) = 1$, and $\widehat{f}(\zeta) = e^{-\pi \zeta^2} = f(\zeta)$.

Example 7.1.8. Let $f(x) = \mathbf{1}_{|x| < R}$ be the characteristic function on the disk $D(0, R)$ in two dimensions.

$$\begin{aligned}\widehat{f}(\zeta) &= \int_{|x| < R} e^{-2\pi i \zeta \cdot x} dx = \int_0^R \int_0^{2\pi} e^{-2\pi i |\zeta| r \cos \theta} d\theta r dr = 2\pi \int_0^R J_0(2\pi |\zeta| r) r dr \\ &= \frac{2\pi}{(2\pi)^2 |\zeta|^2} \int_0^{2\pi |\zeta| R} J_0(r') r' dr' = \frac{R}{|\zeta|} J_1(2\pi |\zeta| R).\end{aligned}$$

The last equality uses the connection formula of Bessel functions, see Section 7.5.1.

7.1.1 Properties of Fourier Transform

There are a few key properties of the Fourier transform. Instead of $\mathfrak{G}_{a,\delta}$, we consider a slightly broader class

$$\mathfrak{F}_\delta := \left\{ f \in C(\mathbb{R}) \mid |f(x)| + |\widehat{f}(x)| \leq \frac{A}{1 + |x|^{1+\delta}} \right\}.$$

Theorem 7.1.9 (Plancherel Identity). Suppose $f \in \mathfrak{F}_\delta$, then

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\widehat{f}(\zeta)|^2 d\zeta.$$

and

$$\int_{-\infty}^{\infty} f(x) \overline{g(x)} dx = \int_{-\infty}^{\infty} \widehat{f}(\zeta) \overline{\widehat{g}(\zeta)} d\zeta.$$

Proof Sketch. We only need to show the 2nd equality. Taking

$$\widehat{f}(\zeta) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x \zeta} dx.$$

Then, by Fubini's theorem,

$$\begin{aligned} \int_{-\infty}^{\infty} \widehat{f}(\zeta)\overline{\widehat{g}(\zeta)}d\zeta &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)e^{-2\pi i x \zeta} dx \overline{\widehat{g}(\zeta)}d\zeta \\ &= \int_{-\infty}^{\infty} f(x) \int_{-\infty}^{\infty} e^{-2\pi i x \zeta} \overline{\widehat{g}(\zeta)}d\zeta dx \\ &= \int_{-\infty}^{\infty} f(x) \overline{\int_{-\infty}^{\infty} e^{2\pi i x \zeta} \widehat{g}(\zeta)d\zeta} dx = \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx. \end{aligned}$$

Moreover, the identity holds for more general cases $f, g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. \square

Another famous property of the Fourier transform is the **uncertainty principle**, meaning localizing in space and time cannot be achieved at the same time. For instance, a localized bump function must have a broad spectrum of frequencies, and a single frequency (i.e., $e^{2\pi i \zeta x}$) must spread over a large domain. Mathematically, it is described by the following.

Theorem 7.1.10 (Uncertainty Principle). Suppose $f, f' \in \mathfrak{F}_\delta$. Define

$$\mathbb{E}_x = \frac{\int_{-\infty}^{\infty} x^2 |f(x)|^2 dx}{\int_{-\infty}^{\infty} |f(x)|^2 dx} \quad \mathbb{E}_\zeta = \frac{\int_{-\infty}^{\infty} \zeta^2 |\widehat{f}(\zeta)|^2 d\zeta}{\int_{-\infty}^{\infty} |\widehat{f}(\zeta)|^2 d\zeta},$$

$$\text{then } E_x E_\zeta \geq \frac{1}{16\pi^2}.$$

Proof. First, by definition, $\widehat{f}'(\zeta) = -2\pi i \zeta \widehat{f}(\zeta)$. Using Plancherel's identity, the inequality is equivalent to

$$\int_{-\infty}^{\infty} x^2 |f(x)|^2 dx \int_{-\infty}^{\infty} |f'(x)|^2 dx \geq \frac{1}{4} \left(\int_{-\infty}^{\infty} |f(x)|^2 dx \right)^2.$$

and by Cauchy-Schwartz Lemma 1.2.2,

$$\begin{aligned} \int_{-\infty}^{\infty} x^2 |f(x)|^2 dx \int_{-\infty}^{\infty} |f'(x)|^2 dx &\geq \left(\int_{-\infty}^{\infty} x f(x) f'(x) dx \right)^2 \\ &= \frac{1}{4} \left(\int_{-\infty}^{\infty} |f(x)|^2 dx \right)^2. \end{aligned}$$

The last step uses integration by parts. □

Theorem 7.1.11 (Convolution). Let $f, g \in \mathfrak{F}_\delta$. Define

$$h(x) = g * f(x) := \int_{-\infty}^{\infty} g(x-y)f(y)dy$$

Then,

$$\widehat{h}(\zeta) = \widehat{f}(\zeta)\widehat{g}(\zeta).$$

Proof Sketch. By Fubini's theorem,

$$\begin{aligned} \int_{-\infty}^{\infty} e^{2\pi i x \zeta} \widehat{f}(\zeta)\widehat{g}(\zeta)d\zeta &= \int_{-\infty}^{\infty} e^{2\pi i x \zeta} \int_{-\infty}^{\infty} f(u)e^{-2\pi i u \zeta} du \widehat{g}(\zeta)d\zeta \\ &= \int_{-\infty}^{\infty} f(u) \int_{-\infty}^{\infty} \widehat{g}(\zeta)e^{2\pi i(x-u)\zeta} d\zeta du \\ &= \int_{-\infty}^{\infty} g(x-u)f(u)du \end{aligned}$$

It also implies $f * g = g * f$. □

This property can be used to deal with convolution-type equations.

Example 7.1.12. Suppose $u \in \mathfrak{F}_\delta$ solves the integral equation

$$\pi \int_{-\infty}^{\infty} e^{-2\pi|x-y|}u(y)dy + \frac{1}{3}u(x) = f(x).$$

Then, after applying the Fourier transform

$$\frac{1}{\zeta^2 + 1}\widehat{u}(\zeta) + \frac{1}{3}\widehat{u}(\zeta) = \widehat{f}(\zeta).$$

Thus, $\widehat{u}(\zeta) = \frac{3(\zeta^2+1)}{\zeta^2+4}\widehat{f}(\zeta)$. The solution is found by applying the inverse Fourier transform.

$$\begin{aligned} u(x) &= 3 \int_{-\infty}^{\infty} \left(1 - \frac{3}{\zeta^2 + 4}\right) \widehat{f}(\zeta)e^{2\pi i x \zeta} d\zeta \\ &= 3f(x) - \frac{9\pi}{2} \int_{-\infty}^{\infty} e^{-4\pi|x-y|}f(y)dy. \end{aligned}$$

7.1.2 Poisson Summation Formula

The Poisson Summation Formula is one of the most interesting properties of the Fourier transform.

Theorem 7.1.13 (Poisson Summation Formula). *If $f \in \mathfrak{G}_{a,\delta}$, then*

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{\zeta \in \mathbb{Z}} \hat{f}(\zeta).$$

The left-hand side can be understood as

$$\lim_{R \rightarrow \infty} \sum_{|n| < R} 2\pi i \operatorname{Res}(F, n)$$

If we can choose a suitable $F(z)$ with simple poles at all integers such that the residue matches $f(n)$, then we can evaluate the left-hand side as a contour integral.

Proof Sketch. A candidate F is $\frac{f(z)}{e^{2\pi iz} - 1}$, the residue at $z = n$ is

$$\int_{D(n,\varepsilon)} \frac{f(z)}{e^{2\pi iz} - 1} dz = \int_{D(0,\varepsilon)} \frac{f(w+n)}{e^{2\pi iw} - 1} dw = \lim_{w \rightarrow 0} w \frac{f(w+n)}{e^{2\pi iw} - 1} = f(n) \frac{1}{2\pi i}.$$

Therefore,

$$\sum_{n \in \mathbb{Z}} f(n) = \lim_{R \rightarrow \infty} \int_{\gamma_R} \frac{f(z)}{e^{2\pi iz} - 1} dz$$

where γ_R is a contour like Figure 7.2 with $R \in \mathbb{Z} + \frac{1}{2}$. On the vertical sides, the magnitude of the denominator is bounded below by one and $f \rightarrow 0$ as $R \rightarrow \infty$. Thus (for any $b > 0$),

$$\sum_{n \in \mathbb{Z}} f(n) = \int_{-\infty - ib}^{\infty - ib} \frac{f(z)}{e^{2\pi iz} - 1} dz - \int_{-\infty + ib}^{\infty + ib} \frac{f(z)}{e^{2\pi iz} - 1} dz.$$

Since $|e^{2\pi iz}| > 1$ in the lower half-plane, and $|e^{2\pi iz}| < 1$ in the upper half-plane, we take the (absolutely) convergent summation

$$\begin{aligned} \frac{1}{e^{2\pi iz} - 1} &= - \sum_{n=0}^{\infty} e^{2n\pi iz} \quad \Im z > 0, \\ \frac{1}{e^{2\pi iz} - 1} &= e^{-2\pi iz} \sum_{n=0}^{\infty} e^{-2n\pi iz} \quad \Im z < 0. \end{aligned}$$

Then

$$\begin{aligned} \int_{-\infty-ib}^{\infty-ib} \frac{f(z)}{e^{2\pi iz} - 1} dz &= \int_{-\infty-ib}^{\infty-ib} \sum_{n=1}^{\infty} f(z) e^{-2n\pi iz} dz \\ &= \sum_{n=1}^{\infty} \int_{-\infty-ib}^{\infty-ib} f(z) e^{-2n\pi iz} dz \quad (\text{why?}) \\ &= \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} f(z) e^{-2n\pi iz} dz = \sum_{n=1}^{\infty} \widehat{f}(-n). \end{aligned}$$

Similarly,

$$- \int_{-\infty+ib}^{\infty+ib} \frac{f(z)}{e^{2\pi iz} - 1} dz = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} f(z) e^{2n\pi iz} dz = \sum_{n=0}^{\infty} \widehat{f}(n).$$

□

Remark 7.1.14. The Poisson Summation Formula can be generalized to the class \mathfrak{F}_δ instead of $\mathfrak{G}_{a,\delta}$. The proof idea is to use the auxiliary function

$$F(x) = \sum_{n \in \mathbb{Z}} f(n+x)$$

which is 1-periodic, that is, $F(1+x) = F(x)$, the bound of f makes sure F exists at every point. For any two points $|x-y| < \varepsilon$,

$$\begin{aligned} |F(x) - F(y)| &\leq \sum_{n \in \mathbb{Z}} |f(n+x) - f(n+y)| \\ &\leq \int_{-\infty}^{\infty} |\widehat{f}(\zeta)| |1 - e^{2\pi i \zeta(x-y)}| d\zeta \\ &\leq 2\pi |x-y| \int_{-R}^R |\widehat{f}(\zeta)| |\zeta| d\zeta + 2 \int_{|\zeta| > R} |\widehat{f}(\zeta)| d\zeta \\ &\leq 2\pi \varepsilon R^2 + \frac{2}{\delta R^\delta} \xrightarrow{R=(\varepsilon\delta)^{-1/(2+\delta)}} C \varepsilon^{\delta/(2+\delta)} \delta^{-2/(2+\delta)}. \end{aligned}$$

which means F is Hölder continuous of order $\delta/(2+\delta)$. Thus, at any $x \in [0, 1)$, $F(x)$ is equal to its Fourier series, see [Jackson \(1930\)](#),

$$F(x) = \sum_{n \in \mathbb{Z}} \widehat{F}(k) e^{2\pi i k x},$$

where the Fourier coefficients

$$\begin{aligned}\widehat{F}(k) &= \int_0^1 e^{-2\pi i k x} F(x) dx = \int_0^1 e^{-2\pi i k x} \sum_{n \in \mathbb{Z}} f(n+x) dx \\ &= \int_{-\infty}^{\infty} f(x) e^{-2\pi i k x} = \widehat{f}(k).\end{aligned}$$

The Poisson Summation Formula is obtained by setting $x = 0$.

The Poisson Summation Formula has many applications in analysis, particularly in estimating *exponential sums*. One interesting example is the earlier work of Van der Corput, calculating the number of lattice points inside a circle. The problem is known as the *Gauss Circle Problem*.

Example 7.1.15 (Gauss Circle Problem). *The Gauss Circle Problem is about estimating the number of lattice points \mathbb{Z}^2 inside a circle of radius R . The quantity of interest is the error $|\text{card}(\mathbb{Z}^2 \cap D(0, R)) - \pi R^2|$. The obvious estimate is $\mathcal{O}(R)$ through a rescaling argument. Van der Corput utilized the Poisson Summation Formula, which transformed the problem into estimating Fourier coefficients. He showed that*

$$|\text{card}(\mathbb{Z}^2 \cap D(0, R)) - \pi R^2| = \mathcal{O}(R^{2/3}).$$

We outline the key idea as follows.

Let χ_R be the characteristic function on $D(0, R)$, that is, $\chi_R(x) = 1$ if $|x| < R$, otherwise zero. It can be transformed into another form:

$$\int_{\mathbb{R}^2} \chi_R(x) dx - \sum_{n \in \mathbb{Z}^2} \chi_R(n) = \widehat{\chi}_R(0) - \sum_{n \in \mathbb{Z}^2} \chi_R(n).$$

Use the calculation in Example 7.1.8,

$$\widehat{\chi}_R(n) = \frac{R}{|n|} J_1(2\pi|n|R).$$

where J_1 denotes the Bessel function of order one. However, we cannot use the Poisson Summation Formula directly due to the nonsmoothness of χ_R . A common way is to use mollification by convolution:

$$\chi_{R,\delta} = \chi_R * \psi_\delta = \int_{\mathbb{R}^2} \frac{1}{\delta^2} \psi\left(\frac{x-y}{\delta}\right) \chi_R(y) dy, \quad \int_{\mathbb{R}^2} \psi(x) dx = 1,$$

where $\text{supp } \psi \subset D(0, 1)$. Then $\widehat{\chi}_{R,\delta}(n) = \widehat{\chi}_R(n) \widehat{\psi}_\delta(n)$ and by Poisson Summation

Formula,

$$\sum_{n \in \mathbb{Z}^2} \chi_{R,\delta}(n) - \pi R^2 = \sum_{n \in \mathbb{Z}^2, |n| \neq 0} \widehat{\chi}_R(n) \widehat{\psi}(\delta n).$$

The extra factor $\widehat{\psi}$ decays fast when $\delta n \gg 1$. Roughly speaking, it plays a role in truncating the right-hand side to $|n|\delta = \mathcal{O}(1)$. Therefore, the estimate on right-hand side is

$$\mathcal{O}(R^{1/2} \sum_{|n| < 1/\delta} |n|^{-3/2}) = \mathcal{O}(R^{1/2} \delta^{-1/2}).$$

Observe that $\chi_{R,\delta}$ and χ_R only differ at the boundary layer of size δ . Thus,

$$\sum_{n \in \mathbb{Z}^2} \chi_{R,\delta}(n) - \sum_{n \in \mathbb{Z}^2} \chi_R(n) = \mathcal{O}(R\delta).$$

The sum of these two errors is $\mathcal{O}(R^{1/2} \delta^{-1/2} + R\delta)$, which is balanced at $\delta = \mathcal{O}(R^{-1/3})$ and the error is at order $\mathcal{O}(R^{2/3})$.

Remark 7.1.16. Nevertheless, this bound is far from the conjectured $\mathcal{O}(R^{1/2+\varepsilon})$. By far, the best known bound is $\mathcal{O}(R^{131/208})$, only a little improvement over a century.

Remark 7.1.17. However, if the circle (it could be any strongly convex smooth shapes) can be randomly placed in the plane, then the **variance** of the error (mean-zero, of course) will be of order $\mathcal{O}(R)$, which suggests the conjectured bound is likely true when the circle is randomly placed.

7.1.3 Radon Transform

In this section, we introduce the two-dimensional Radon transform. Suppose f is compactly supported, the Radon transform of f is

$$\mathcal{R}f(\theta, s) = \int_{-\infty}^{\infty} f(z \sin \theta + s \cos \theta, -z \cos \theta + s \sin \theta) dz$$

It describes an integral of f on the line $\{(x, y) : x \cos \theta + y \sin \theta = s\}$. If we denote $v = (\cos \theta, \sin \theta)$ and $v^\perp = (\sin \theta, -\cos \theta)$, then

$$\mathcal{R}f(\theta, s) = \int_{-\infty}^{\infty} f(sv + zv^\perp) dz.$$

An important fact about the Radon transform is the connection with the Fourier transform.

Theorem 7.1.18 (Fourier Slice Theorem). *TODO*

7.2 Laplace Transform

TODO

7.3 Melin Transform

TODO

7.4 Gamma Functions

The Gamma function is defined (by Legendre) as follows.

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt.$$

This function exists on $\Re(z) > 0$ (otherwise not integrable) and is differentiable

$$\Gamma'(z) = \int_0^{\infty} e^{-t} t^{z-1} \ln t dt,$$

which exists (and is continuous) on the same region. Thus $\Gamma(z)$ is holomorphic on $\Re(z) > 0$. Using integration by parts, if $\Re(z) > 1$,

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt = -e^{-t} t^{z-1} \Big|_0^{\infty} + (z-1) \int_0^{\infty} e^{-t} t^{z-2} dt = (z-1)\Gamma(z-1).$$

Since $\Gamma(1) = 1$. The special values on positive integers can be calculated inductively.

Example 7.4.1. Suppose $n \in \mathbb{N}$, then $\Gamma(n+1) = n!$.

7.4.1 Identities

We also notice that for any non-integer values of $z \in (n, n+1)$,

$$\Gamma(z) = (z-1) \cdots (z-n) \Gamma(z-n).$$

Thus, the values of $\Gamma(z)$, $z \in (0, 1)$ are essential to compute other values.

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Theorem 7.4.2 (Euler's reflection formula). *If $z \in (0, 1)$, then*

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}.$$

Proof Sketch. The product is equivalent to

$$\begin{aligned} \Gamma(z)\Gamma(1-z) &= \int_0^\infty e^{-t}t^{z-1}dt \int_0^\infty e^{-s}s^{-z}ds \stackrel{s=ut}{=} \int_0^\infty \int_0^\infty e^{-(tu+t)}(u)^{-z} dtdu \\ &= \int_0^\infty \frac{u^{-z}}{1+u} du \stackrel{u=e^v}{=} \int_{-\infty}^\infty \frac{e^{v(1-z)}}{1+e^v} dv. \end{aligned}$$

Using the rectangle contour in the upper half-plane, the integral becomes

$$\int_{-\infty}^\infty \frac{e^{v(1-z)}}{1+e^v} dv - \int_{-\infty+2\pi i}^{\infty+2\pi i} \frac{e^{v(1-z)}}{1+e^v} dv = -2\pi i e^{\pi i(1-z)} = 2\pi i e^{-z\pi i}$$

Since

$$\int_{-\infty+2\pi i}^{\infty+2\pi i} \frac{e^{v(1-z)}}{1+e^v} dv = e^{2\pi i(1-z)} \int_{-\infty}^\infty \frac{e^{v(1-z)}}{1+e^v} dv,$$

Therefore,

$$\Gamma(z)\Gamma(1-z) = \frac{2\pi i e^{-z\pi i}}{1 - e^{-2\pi iz}} = \frac{2\pi \sin(\pi z)}{1 - \cos(2\pi z)} = \frac{\pi}{\sin(\pi z)}.$$

□

The reflection formula extends Γ to $\mathbb{C} \setminus \mathbb{Z}$ besides the positive integers, see Figure 7.3. Immediately, we find the value of $\Gamma(1/2) = \sqrt{\pi}$ and can be used to evaluate Γ at half-integers. A similar change of variable in the above proof leads to the following identity.

Theorem 7.4.3 (Beta Function).

$$B(p, q) := \int_0^1 t^{p-1}(1-t)^{q-1} dt = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \quad p, q > 0.$$

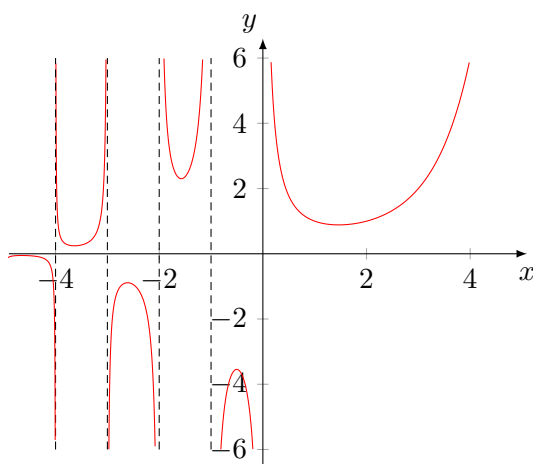


Figure 7.3: Plot of Gamma function.

Proof. It is quite straightforward.

$$\begin{aligned}
 \Gamma(p)\Gamma(q) &= \int_0^\infty \int_0^\infty e^{-t(1+v)} t^{p+q-1} v^{q-1} dt dv \\
 &\stackrel{s=t(1+v)}{=} \int_0^\infty \int_0^\infty e^{-s} s^{p+q-1} (1+v)^{-(p+q)} v^{q-1} ds dv \\
 &= \Gamma(p+q) \int_0^\infty \frac{v^{q-1}}{(1+v)^{p+q}} dv \\
 &\stackrel{t=1/(1+v)}{=} \Gamma(p+q) \int_0^1 (1-t)^{q-1} t^{p-1} dt.
 \end{aligned}$$

□

This implies some interesting integrals like

$$\int_0^{\pi/2} \sin^{2p-1} x \cos^{2q-1} x dx = \frac{B(p, q)}{2}.$$

Example 7.4.4. $\int_0^{\pi/2} \sin^m x dx = \frac{B(\frac{m+1}{2}, \frac{1}{2})}{2} = \frac{\sqrt{\pi}}{2} \Gamma(\frac{m+1}{2}) / \Gamma(\frac{m+2}{2}).$

Example 7.4.5. $\int_{-1}^1 (1-t^2)^{n-\frac{1}{2}} dt = 2 \int_0^{\pi/2} \cos^{2n} x dx = \sqrt{\pi} \Gamma(n+1/2) / \Gamma(n+1).$

7.4.2 Asymptotic Behavior

The asymptotic expansion of $\Gamma(z)$ can be derived by **Laplace's method**.

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt = \int_0^\infty e^{-t+(z-1)\ln t} dt \stackrel{t=zu}{=} e^{-z} z^z \int_0^\infty e^{-zf(u)} du.$$

where $f(u) = u - \ln u - 1$, its minimum is achieved at $u = 1$ and $f''(1) = \frac{1}{u^2} = 1$. By the Laplace's method (see Section 8.2.1, the leading term is

$$\Gamma(z) \sim \sqrt{2\pi} e^{-z} z^{z-1/2}.$$

This asymptotic expression is also called *Stirling formula*. It implies the *Stirling approximation* as follows, see Figure 7.4.

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n.$$

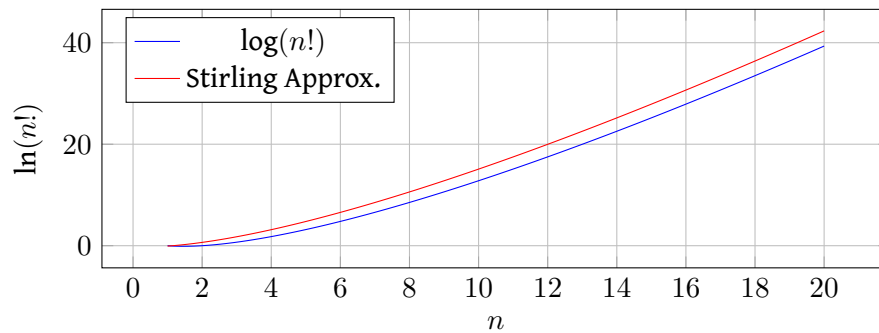


Figure 7.4: $n!$ vs. Stirling Approximation.

7.5 Bessel Functions

The Bessel functions are closely related to the Helmholtz equation in polar coordinates (two dimensions).

$$\Delta u + k^2 u = 0.$$

Let $x = \rho \cos \theta$, $y = \rho \sin \theta$, then

$$\Delta = (\partial_\rho \cos \theta - \sin \theta \frac{1}{\rho} \partial_\theta)^2 + (\partial_\rho \sin \theta + \cos \theta \frac{1}{\rho} \partial_\theta)^2 = \partial_\rho^2 + \frac{1}{\rho} \partial_\rho + \frac{1}{\rho^2} \partial_\theta^2$$

Using separation of variables $u = y(\rho)\phi(\theta)$, then

$$\frac{\rho^2 y'' + \rho y' + \rho^2 k^2 y}{y} = -\frac{\phi''}{\phi} = \nu^2,$$

where ν^2 denotes the eigenvalues of $\phi'' + \nu^2\phi = 0$ with periodic boundary condition, which implies $\nu \in \mathbb{N}$. Therefore,

$$\rho^2 y'' + \rho y' + (k^2 \rho^2 - \nu^2)y = 0.$$

The **Bessel equation** is given by setting $z = k\rho$:

$$z^2 y'' + zy' + (z^2 - \nu^2)y = 0.$$

Here, this equation does not require ν to be an integer. The series solution can be found by the **method of Frobenius**, which seeks the representation

$$y(z) = z^p \sum_{k=0}^{\infty} c_k z^k.$$

The *indicial equation* $p^2 = \nu^2$ leads to two cases $p = \pm\nu$ and

$$c_n = -\frac{c_{n-2}}{(n+p)^2 - \nu^2}.$$

The starting coefficients can be arbitrary. Let $c_1 = 0$, we obtain the following definitions for *Bessel functions of first kind* (up to a scaling):

1. $p = \nu \geq 0$, the resulting series is

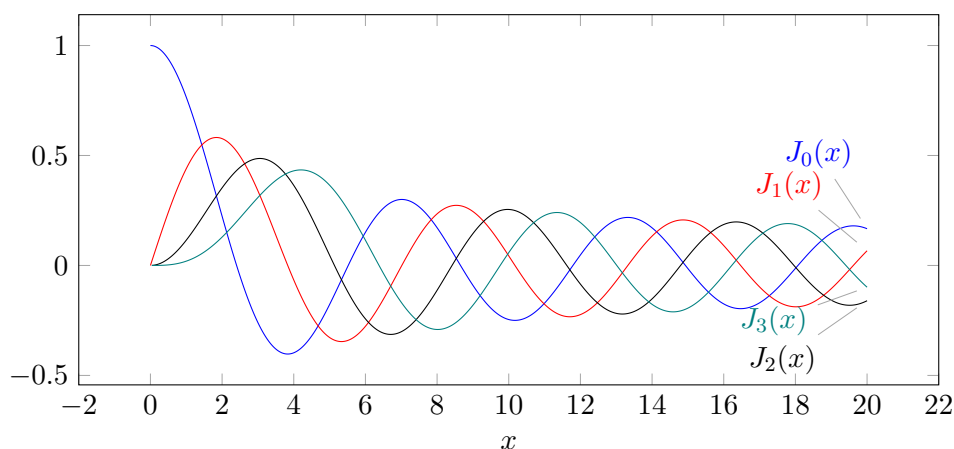
$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(n + \nu + 1)} \left(\frac{z}{2}\right)^{2n+\nu}$$

This function is holomorphic for $|\arg z| < \pi$. The series is convergent on $0 < |z| < \infty$ (why?). When $\nu \in \mathbb{N}$, it is holomorphic on \mathbb{C} , since the factor z^ν does not have a branching point at $z = 0$ anymore. For $z \in \mathbb{R}^+$, see Figure 7.5 for the plots of J_ν .

2. $p = -\nu < 0$, the resulting series is

$$J_{-\nu}(z) = \sum_{n=\nu}^{\infty} \frac{(-1)^n}{n! \Gamma(n - \nu + 1)} \left(\frac{z}{2}\right)^{2n-\nu}.$$

If $\nu \in \mathbb{N}$, we find $J_{-\nu} = (-1)^\nu J_\nu$, meaning the two solutions are linearly dependent. In that case, the Bessel function of the second kind is introduced

Figure 7.5: Plots of Bessel functions J_ν .

by

$$Y_\nu(z) = \frac{J_\nu \cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)}.$$

This note will focus on the first kind since most properties can be re-derived for Y_ν from the above definition.

7.5.1 Bessel Integral

From the series solution, we can derive the connection formula

$$z^\nu \frac{d}{dz} (z^{-\nu} J_\nu(z)) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n! \Gamma(n + \nu + 2)} \left(\frac{z}{2}\right)^{2n+\nu+1} = -J_{\nu+1}(z).$$

Similarly (or flip $\nu \rightarrow -\nu$), we obtain

$$z^{-\nu} \frac{d}{dz} (z^\nu J_\nu(z)) = J_{\nu-1}(z)$$

Therefore, the recurrence relations are:

$$\begin{aligned} \frac{2\nu}{z} J_\nu(z) &= J_{\nu-1}(z) + J_{\nu+1}(z), \\ 2J'_\nu(z) &= J_{\nu-1}(z) - J_{\nu+1}(z). \end{aligned}$$

The recurrence relations are beneficial when dealing with the generating function,

$$g(z, t) = \sum_{n=-\infty}^{\infty} J_n(z) t^n.$$

Then,

$$\begin{aligned}\partial_z g(z, t) &= \sum_{n=-\infty}^{\infty} J'_n(z) t^n = \frac{1}{2} \sum_{n=-\infty}^{\infty} (J_{n-1}(z) + J_{n+1}(z)) t^n \\ &= \frac{1}{2} (t - t^{-1}) g(z, t).\end{aligned}$$

The solution is $g(z, t) = e^{\frac{1}{2}(t-t^{-1})z}$ (why?). The Bessel functions $J_n(z)$ are the coefficients of the Laurent series of $g(z, t)$ following any contour of t around the origin.

$$J_n(z) = \frac{1}{2\pi i} \int_{\gamma} e^{\frac{1}{2}(t-t^{-1})z} t^{-(n+1)} dt.$$

When the contour is a circle $t = e^{i\theta}$,

$$J_n(z) = \frac{1}{2\pi} \int_0^{2\pi} e^{iz \sin \theta} e^{-in\theta} d\theta = \frac{1}{\pi} \int_0^{\pi} \cos(z \sin \theta - n\theta) d\theta.$$

Another useful representation is the *Poisson type formula*. We introduce the auxiliary function:

$$\psi(z) = z^n \int_{-1}^1 (1-t^2)^{n-\frac{1}{2}} e^{izt} dt, \quad n > -\frac{1}{2}.$$

The reason is the following lemma.

Lemma 7.5.1. $\psi(z)$ is a solution to the Bessel equation of order n .

Proof. We bring ψ into the Bessel equation, and we need to deal with

$$(z^2 \partial_z^2 + z \partial_z + (z^2 - n^2)) z^n.$$

where z^n is treated a multiplicative operator. Then

$$z \partial_z z^n = z^{n+1} \partial_z + n z^n, \quad z^2 \partial_z^2 z^n = z^{n+2} \partial_z^2 + 2n z^{n+1} \partial_z + n(n-1) z^n$$

We verify that

$$\begin{aligned} & \underbrace{z^{-n-1}(z^2\partial_z^2 + z\partial_z + (z^2 - n^2))z^n}_{=z\partial_z^2+(1+2n)\partial_z+z} \int_{-1}^1 (1-t^2)^{n-\frac{1}{2}} e^{izt} dt \\ &= \int_{-1}^1 (1-t^2)^{n-\frac{1}{2}} (z(1-t^2) + (1+2n)it) e^{izt} dt \\ &= -i \int_{-1}^1 \frac{\partial}{\partial t} \left((1-t^2)^{n+\frac{1}{2}} e^{izt} \right) dt = 0. \end{aligned}$$

□

Notice that the integral (check it with Theorem 7.4.3)

$$\int_{-1}^1 (1-t^2)^{n-\frac{1}{2}} dt = \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} \sqrt{\pi}.$$

Then, compare $\psi(z)$ with $J_n(z)$ near $z \rightarrow 0$, we find the correct scaling for ψ :

$$\begin{aligned} J_n(z) &= \frac{\left(\frac{z}{2}\right)^n}{\sqrt{\pi}\Gamma(n+\frac{1}{2})} \int_{-1}^1 (1-t^2)^{n-\frac{1}{2}} e^{izt} dt \\ &= \frac{\left(\frac{z}{2}\right)^n}{\sqrt{\pi}\Gamma(n+\frac{1}{2})} \int_0^\pi \sin^{2n} \theta e^{iz \cos \theta} d\theta. \end{aligned}$$

7.5.2 Asymptotic Behavior

The asymptotic behavior can be analyzed by the stationary phase method.

$$J_n(z) = \frac{1}{\pi} \Re \int_0^\pi e^{iz \sin \theta} e^{-in\theta} d\theta$$

The stationary point of $\phi(\theta) = z \sin \theta - n\theta$ is at $z \cos \theta^* = n$ or $\theta^* = \arccos(n/z) = \frac{\pi}{2} - \mathcal{O}(z^{-1})$ and $\phi''(\theta^*) = -z \sin \theta^* = -z \sqrt{1 - n^2/z^2} < 0$. It means locally

$$\phi(\theta) = \phi(\theta^*) + \frac{1}{2} \phi''(\theta^*) (\theta - \theta^*)^2 + \dots$$

Then (see Chapter 8.2.3)

$$\begin{aligned} \int_0^\pi e^{iz \sin \theta} e^{-in\theta} d\theta &\sim \int_{\theta^*-\varepsilon}^{\theta^*+\varepsilon} e^{-i\phi''(\theta^*)(\theta-\theta^*)^2} d\theta \sim e^{i\phi(\theta^*)} \sqrt{\frac{2\pi}{|\phi''(\theta^*)|}} e^{-i\frac{\pi}{4}} \\ &\sim \exp\left(i\left(z - n\frac{\pi}{2} - \frac{\pi}{4}\right)\right) \sqrt{\frac{2\pi}{z}} + \mathcal{O}(z^{-1}). \end{aligned}$$

Therefore, $J_n(z) \sim \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{n\pi}{2} - \frac{\pi}{4}\right)$.

7.6 Exercises

 **Problem 7.6.1.**

Extended Reading

Jackson, D. (1930). *The theory of approximation*, volume 11. American Mathematical Soc.