


# Chapter 4

## Differential Operators

*If there is a problem you can't solve, then there is an easier problem you can solve: find it.*

– George Pólya

 HE study of differential equations often begins with classical notions of differentiation and smooth functions. However, many physically meaningful problems involve irregular data, singular sources, or solutions that lack the smoothness required by classical analysis. To rigorously address such concerns, modern mathematical analysis must extend the concept of differentiation well beyond its traditional boundaries.

## 4.1 Distributions

*Distribution* was introduced by Laurent Schwartz in 1940s. A distribution can be viewed as an extension of a function.

### 4.1.1 Motivation

Let  $u(x) \in C(\mathbb{R})$  be a continuous function. The function can be uniquely determined by evaluating at each point  $\{u(x) \mid x \in \mathbb{R}\}$ , which treats the function as an ordered set or a geometric curve. If, in addition, the function  $u(x)$  is  $2\pi$ -periodic, the function can be alternatively determined by its Fourier coefficients,

$$u(x) = \sum_{k \in \mathbb{Z}} u_k e^{ikx},$$

since the Fourier series converges pointwise at continuous points. It implies that the set

$$\left\{ \int_0^{2\pi} u(x) e^{ikx} dx \right\}_{k \in \mathbb{Z}}$$

determines the periodic continuous function  $u(x)$  uniquely.

If we replace the Fourier modes with general functions  $\psi \in S$ , the set  $S$  is known as *test functions*, and we can ask whether or not the inner product

$$\left\{ \int_{\mathbb{R}} \psi(x) u(x) dx \right\}, \quad \psi \in S$$

can uniquely determine  $u(x)$ . In particular, to make the integral valid, we may assume  $S$  is the set of smooth functions with compact support,  $S = C_0^\infty(\mathbb{R})$ .

**Lemma 4.1.1.** *Let  $u \in C(\mathbb{R})$ , then the set of values*

$$\left\{ \int_{\mathbb{R}} \psi(x) u(x) dx \right\}, \quad \psi \in C_0^\infty(\mathbb{R})$$

*uniquely determines  $u$ .*

*Proof.* If there are two distinct functions  $u_1, u_2 \in C(\mathbb{R})$  satisfying the requirement, then there exists an open set  $\mathcal{I} \subset \mathbb{R}$  that  $u_1 > u_2$ , then one can take  $\psi = \chi$ , where  $\chi$  is a bump function with small compact support on  $\mathcal{I}$  such that  $\langle \psi, u_1 - u_2 \rangle > 0$ . Contradiction.  $\square$

In fact, one can find  $u(x)$  in an explicit way.

$$u(x) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{\mathbb{R}} \psi \left( \frac{x-y}{\varepsilon} \right) u(y) dy,$$

where  $\psi$  is a bump function from  $C_0^\infty$  satisfying the normalization. The functions in  $C_0^\infty$  are corresponding to numbers  $L_\psi \in \mathbb{R}$  for some  $u$ :

$$L_\psi = \int_{\mathbb{R}} \psi(x) u(x) dx.$$

This derives the following necessary property for  $L_\psi$ .

**Condition 4.1.2.** The numbers  $L_\psi$  satisfy linearity, that is,

$$L_{a\psi+b\phi} = aL_\psi + bL_\phi.$$

If  $\psi_n \in C_0^\infty$  converges to  $\psi \in C_0^\infty$  uniformly for any derivatives and shares the same support  $[-A, A]$  (we denote this by  $\psi_n \xrightarrow{uc} \psi$ ), then

$$|L_{\psi_n} - L_\psi| \leq \int_{[-A, A]} |\psi_n(x) - \psi(x)| u(x) dx \rightarrow 0.$$

This leads to another necessary property for  $L_\psi$ .

**Condition 4.1.3.**  $L_{\psi_n} \rightarrow L_\psi$  if  $\psi_n \xrightarrow{uc} \psi$ .

However, the condition that  $L_{\psi_n} \rightarrow L_\psi$  if  $\psi_n \xrightarrow{uc} \psi$  does not mean there exists a function  $u \in C(\mathbb{R})$ , the other direction is correct.

**Example 4.1.4.** Suppose  $L_{\psi_n} u = \psi_n(0)$ , then we cannot find the continuous function  $u$  (why?).

### 4.1.2 Functions to Distributions

If we do not restrict  $u \in C(\mathbb{R})$ , then we wonder what Conditions 4.1.2 and 4.1.3 bring to us. Since we cannot say the existence of some function  $u$ , we call the set  $\ell := \{L_\psi\}$ ,  $\psi \in C_0^\infty(\mathbb{R})$ , as a distribution.

Symbolically, we can view  $\ell$  itself as an operator (linear functional) on  $\psi \in \mathcal{D} = C_0^\infty(\mathbb{R})$  that

$$\ell(\psi) := L_\psi := \langle \ell(x), \psi(x) \rangle.$$

The notation  $\ell(x)$  is just a symbol, not necessarily a function evaluated at  $x$ . The distributions  $\ell$  form a vector space (dual space), denoted by  $\mathcal{D}'$ .

**Example 4.1.5.** *The previous lemma implies that  $C(\mathbb{R}) \subset \mathcal{D}'$ .*

**Example 4.1.6 (Dirac).** *For instance,  $\ell(\psi) = \psi(0)$  defines a distribution  $\ell$ , called Dirac  $\delta$ -function,  $\ell(x) = \delta(x)$ .*

**Example 4.1.7 (Heaviside).** *Let  $\ell(\psi) = \int_0^\infty \psi(x)dx$ . Then,  $\ell(x)$  is a step-shape function, known as the **Heaviside function**,*

$$\ell(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases}$$

*This function is not continuous, and the value of  $\ell(0)$  can be arbitrary.*

**Example 4.1.8.** *Similarly,  $\ell(\psi) = \psi^{(k)}$  also defines a distribution, and it corresponds to  $(-1)^k \delta^{(k)}(x)$ .*

### 4.1.3 Properties of Distributions

Besides the linearity of the vector space formed by distributions, the product of a distribution and a smooth function is still a distribution.

**Definition 4.1.9.** *Let  $\ell \in \mathcal{D}'$ , and  $g \in C^\infty(\mathbb{R})$ , the product  $g\ell$  (or  $\ell g$ ) is defined by*

$$g\ell(\psi) = \langle g\ell, \psi \rangle := \langle \ell, g\psi \rangle.$$

*Thus,  $g\ell \in \mathcal{D}'$ .*

**Remark 4.1.10.** *However, if  $g$  itself is also a distribution too, the product of two distributions is not a distribution. We cannot define this structure through the two conditions mentioned in the last section.*

**Definition 4.1.11.** *A shift to the right by  $a$  of the distribution  $\ell(x)$  is denoted by  $\ell_{\cdot-a}$  (these are just symbols)*

$$\ell_{\cdot-a}(\psi) := \langle \ell(x-a), \psi(x) \rangle = \langle \ell(x), \psi(x+a) \rangle$$

**Definition 4.1.12.** The rescaling of a distribution, denoted by  $\ell_k$ , is defined by

$$\ell_k(\psi) := \langle \ell(kx), \psi(x) \rangle = \frac{1}{|k|} \langle \ell(x), \psi\left(\frac{x}{k}\right) \rangle.$$

**Example 4.1.13.** Recall  $\delta(x)$  is the Dirac function, then

$$\delta_{-a}(\psi) = \psi(a), \quad \delta_k(\psi) = \frac{1}{|k|} \psi(0).$$

Sometimes we can see  $\int_{-\infty}^{\infty} \delta(x) dx = 1$ , it actually means  $\delta(1) = \langle \delta, 1 \rangle = 1$ . In fact, the integral can be an arbitrarily small open set around the origin,

$$\int_{-a}^a \delta(kx) dx = \frac{1}{|k|}, \quad \forall a > 0.$$

and

$$\int_a^b \delta(x - c) dx = \begin{cases} 1, & c \in (a, b), \\ 0, & c \notin (a, b). \end{cases}$$

Note that we cannot evaluate the integral above uniquely when  $c = a$  or  $c = b$  without imposing additional rules. It means  $\delta$  is applied to a discontinuous function.

**Definition 4.1.14 (Convergence in distribution).** Let  $\ell_n \in \mathcal{D}'$  be a sequence of distributions that converge **weakly** to  $\ell \in \mathcal{D}'$  as  $n \rightarrow \infty$ , it means

$$\ell_n(\psi) \rightarrow \ell(\psi), \quad \forall \psi \in \mathcal{D}, \quad n \rightarrow \infty.$$

And we denote  $\ell_n \xrightarrow{w} \ell$ .

**Example 4.1.15.** Let  $\ell_k(x) = k^2 \sin(kx) \in \mathcal{D}'$ , then  $\ell_k \xrightarrow{w} 0$ .

$$\begin{aligned} \langle \ell_k(x), \psi(x) \rangle &= \int_{-A}^A k^2 \sin(kx) \psi(x) dx \\ &= -k \cos(kx) \psi(x) \Big|_{-A}^A + \int_{-A}^A k \cos(kx) \psi'(x) dx \\ &= \sin(kx) \psi'(x) \Big|_{-A}^A - \int_{-A}^A \sin(kx) \psi''(x) dx \\ &\rightarrow 0. \end{aligned}$$

This is the Riemann-Lebesgue theorem.

**Example 4.1.16.** Let  $\ell_n(x)$  be

$$\ell_n(x) = \begin{cases} n, & x \in [0, \frac{1}{n}], \\ 0, & x \notin [0, \frac{1}{n}]. \end{cases}$$

Then  $\ell_n \xrightarrow{w} \delta$ .

**Example 4.1.17.** Consider the principle value of  $\frac{1}{x}$ , which defines the following map

$$\text{p. v.} \left( \frac{1}{x} \right) : \phi \rightarrow \lim_{\varepsilon \rightarrow 0^+} \int_{|x| > \varepsilon} \frac{\phi(x)}{x} dx.$$

It defines a distribution on  $\mathbb{R}$ . The integral operators

$$\mathcal{I}_n(\phi) = \int_{|x| > \frac{1}{n}} \frac{\phi(x)}{x} dx$$

converges to p. v.  $\left( \frac{1}{x} \right)$  in distribution.

#### 4.1.4 Derivatives of Distributions

The derivative of a distribution is defined via integration by parts, which shifts differentiation to the test functions.

**Definition 4.1.18.** Let  $\ell$  be a distribution, then

$$\ell'(\psi) := \langle \ell'(x), \psi(x) \rangle := -\langle \ell(x), \psi'(x) \rangle.$$

It immediately follows that the derivative of any order of a distribution remains a distribution (by induction). If  $\ell$  itself is a differentiable function, then  $\ell'$  is its usual derivative.

**Example 4.1.19.** If  $\ell$  is the Heaviside function, then  $\ell' = \delta$  (why?).

**Lemma 4.1.20.** The differentiation operator for distribution is continuous, that is,

$$\ell_n \xrightarrow{w} \ell \quad \text{implies} \quad \ell'_n \xrightarrow{w} \ell'.$$

*Proof.* The term “continuity” refers to the weak topology. The theorem is easy using the definition.  $\square$

**Example 4.1.21.** *The finite difference scheme*

$$\varepsilon^{-1}(\ell_{+\varepsilon} - \ell) \xrightarrow{w} \ell', \quad \varepsilon \rightarrow 0.$$

By definition, we can easily derive the product rule for differentiation.

**Lemma 4.1.22.** *If  $g \in C^\infty(\mathbb{R})$ , then the product rule applies.*

$$(g\ell)' = g'\ell + g\ell'.$$

**Example 4.1.23.** *Consider  $(H(x) \frac{\sin kx}{k})''$ ,  $k \neq 0$ , where  $H(x)$  is the Heaviside function.*

$$(H(x) \frac{\sin kx}{k})' = H'(x) \frac{\sin kx}{k} + H(x) \cos(kx).$$

Since  $H' = \delta$ , and  $H'(x) \frac{\sin kx}{k} = 0$  (why?). Then,

$$(H(x) \cos kx)' = H'(x) \cos kx - H(x)k \sin(kx) = \delta(x) - H(x)k \sin(kx).$$

In other words,

$$\left( \frac{d^2}{dx^2} + k^2 \right) \left( H(x) \frac{\sin kx}{k} \right) = \delta(x).$$

**Remark 4.1.24.** *The multivariate distribution can be defined similarly, including the partial derivatives.*

## 4.2 Distributions of Compact Support

The previous sections have mentioned the Dirac  $\delta$ -function and its derivatives. It is straightforward to see that

$$\int_{\mathbb{R}} \delta^{(k)}(x) \phi(x) dx = 0,$$

if  $0 \notin \text{supp } \phi$ . This implies that  $\delta^{(k)}$  is compactly supported (at  $x = 0$ ). This raises a question: if  $\ell \in \mathcal{D}'$  is compactly supported on a point, what else can  $\ell$  be?

Recall that a distribution  $\ell$  satisfies the condition 4.1.3, which implies some “continuity”, that is, if given any  $\alpha$ ,  $\partial^\alpha \phi_j \rightarrow \partial^\alpha \phi_j$  for all  $x \in K \subset\subset \Omega$  uniformly, then

$$\ell(\phi_j) \rightarrow \ell(\phi).$$

The continuity can be quantified through the following metric (not a norm).

**Remark 4.2.1.** *Intuitively, we can assign  $C^k(K)$  norm to  $C^\infty$  functions with  $k \rightarrow \infty$ ,*

that is

$$\|f\| = \sum_{|\alpha| \geq 0} \|\partial^\alpha f\|_{C(K)}.$$

Each norm in the sum is a standard norm for continuous functions. However, this does not define a norm since  $\|f\|$  is not always finite for  $f \in C^\infty$  functions.

**Definition 4.2.2.** Let  $K_1 \subset K_2 \subset \dots$  be a sequence of compact subsets such that

$$\bigcup_{j \geq 1} \text{int}(K_j) = \Omega.$$

The following function defines a metric on  $C^\infty(\Omega)$ :

$$d_C(f, g) = \sum_{j=1, m=1}^{\infty} 2^{-j-m} \frac{\|f - g\|_{K_j, m}}{1 + \|f - g\|_{K_j, m}},$$

where  $\|f\|_{K, m} = \sup_{x \in K, |\alpha| \leq m} |\partial^\alpha f(x)|$ .

**Remark 4.2.3.** One can show  $d_C$  is indeed a metric on  $C^\infty(\Omega)$ . This induces a topology on  $C^\infty(\Omega)$  which allows us to define the desired “continuity”.

**Theorem 4.2.4.**  $C^\infty(\Omega)$  is complete under the metric  $d_C$ .

*Proof.* Let  $\phi_k \in C^\infty(\Omega)$  be a Cauchy sequence under  $d_C$ . Then  $\partial^\alpha \phi_k$  is a Cauchy sequence under  $C(K_j)$  norm. Thus (since  $C(K_j)$  is complete), there exists  $\phi_\alpha \in C(K_j)$  that

$$\partial^\alpha \phi_j \rightarrow \phi_\alpha \in C(K_j).$$

Thus,  $\phi_\alpha = \partial^\alpha \phi_0$ , where  $\phi_0 = \lim_{k \rightarrow \infty} \phi_k$ , which implies  $\phi_k \rightarrow \phi$  in  $C^\infty(K_j)$  and let  $j \rightarrow \infty$ .  $\square$

**Theorem 4.2.5.** Let  $\ell \in \mathcal{D}'$  be compactly supported on  $\Omega$ . Then, there exist  $K, m$  and a constant  $C$  (depends on  $\ell$ ) that

$$|\ell(\phi)| \leq C \|\phi\|_{K, m}.$$

The smallest possible  $m$  is called the order of  $\ell$ .

*Proof.* First, we show that  $\ell \in \mathcal{D}'$  can be extended to a linear functional on  $C^\infty(\Omega)$ . Let  $K = \text{supp}(\ell) \subset \Omega$ , we take  $\chi \in C_0^\infty(\Omega)$  such that  $\chi = 1$  on  $K$ . Then,

$$\ell(\phi) := \langle \ell, \phi \rangle = \langle \ell, \chi\phi \rangle.$$

Thus  $\ell : \phi \rightarrow \langle \ell, \chi\phi \rangle$  is a linear functional for  $C^\infty(\Omega)$ .

By the continuity of  $\ell$ , there exists  $\delta = \delta(\varepsilon) > 0$  that if  $d_C(\chi\phi, 0) < \delta$ ,  $|\ell(\phi)| < \varepsilon$ . There exists  $N$  and  $j_N$  that if  $\|\chi\phi\|_{K_N, N} < \delta'$  for some  $\delta'$ , then

$$d_C(\chi\phi, 0) = \sum_{j=1, m=1}^{\infty} 2^{-j-m} \frac{\|\chi\phi\|_{K_j, m}}{1 + \|\chi\phi\|_{K_j, m}} < \delta.$$

Therefore,  $|\ell(\phi)| < \varepsilon$  whenever  $\|\chi\phi\|_{K_N, N} < \delta'$ , which implies

$$|\ell(\phi)| \leq \frac{\varepsilon}{\delta'} \|\chi\phi\|_{K_N, N} \leq C_\chi \frac{\varepsilon}{\delta'} \|\phi\|_{K_N, N}.$$

□

**Corollary 4.2.6 (Hörmander (2003)).** *If  $\ell \in \mathcal{D}'$  is compactly supported, then  $\ell$  has a finite order.*

**Example 4.2.7.** *The distribution  $\delta^{(k)}(x)$  has an order of  $k$ .*

**Example 4.2.8.** *The distribution p. v.  $(\frac{1}{x})$  has an order of 1 (why?).*

The following theorem can be easily extended to high dimensions.

**Theorem 4.2.9.** *If  $\ell \in \mathcal{D}'(\mathbb{R})$  is supported at a finite number of points, denoted by  $P$ , then  $u$  is a finite linear combination of*

$$\ell(x) = \sum_{p \in P} \sum_{\alpha=0}^N c_{\alpha, p} \partial^\alpha \delta(x - p).$$

*Proof.* It suffices to show for a singleton set  $P = \{0\}$ . Assume  $\ell$  has an order of  $m$  and

$$|\ell(\phi)| \leq C \|\phi\|_{K, m}$$

For any  $\phi = \Phi + \Psi$  decomposed as its  $m$ -th Taylor polynomial and the remainder.

$$\Psi(x) = \phi(x) - \underbrace{\sum_{k=0}^m \phi^{(k)}(0) \frac{x^k}{k!}}_{=\Phi}.$$

We have the estimate

$$\|\partial^\alpha \Psi(x)\|_{C(K)} \leq \frac{1}{(m+1-\alpha)!} \|\phi\|_{K, m+1} |x|^{m+1-\alpha}.$$

Then take any bump function  $\eta \in C_0^\infty(\mathbb{R})$  that equals to 1 in  $B_0(1)$ , we have (for any  $\varepsilon > 0$ )

$$\begin{aligned} |\ell(\Psi)| &= |\langle \ell, \eta(x/\varepsilon)\Psi \rangle| \leq C \|\eta(x/\varepsilon)\Psi\|_{K, m} \\ &= C \sup_{x \in B_0(\varepsilon)} \sup_{|\alpha| \leq m} |\partial^\alpha (\eta(x/\varepsilon)\Psi)| \\ &\leq [C(\eta, m) \|\phi\|_{K, m+1}] \cdot \varepsilon \rightarrow 0, \end{aligned}$$

as  $\varepsilon \rightarrow 0$ . Therefore,  $\ell(\phi) = \ell(\Phi)$ . Finally,  $\ell$  is uniquely determined by monomials, and  $\ell(x^n) = t_n$  means  $\ell(x) = \sum_{n=0}^m \frac{t_n}{n!} (-1)^n \delta^{(n)}(x)$ .  $\square$

**Example 4.2.10.** For instance, if  $p(x)u = 0$  for all  $x \in \mathbb{R}$ , where  $p(x) = \prod_{j=1}^N (x - x_j)$  is a polynomial with no repeated roots. Then  $P = \{x_j\}_{j=1}^N$ , and

$$u(x) = \sum_{j=1}^N \sum_{\alpha=0}^m c_{j, \alpha} \delta^{(\alpha)}(x - x_j).$$

By testing  $p(x)u$  against  $(x - x_j)^\beta$  (locally), we find that  $c_{j, \alpha} = 0$  for all  $\alpha \geq 1$ . Thus,

$$u(x) = \sum_{j=1}^N c_j \delta(x - x_j).$$

### 4.3 Green's Functions

We consider the linear ordinary differential equation of constant coefficients

$$\mathcal{A}u = 0, \quad \mathcal{A} = \sum_{k=0}^n a_k \frac{d^k}{dx^k}.$$

**Definition 4.3.1.** The fundamental solution of the operator  $\mathcal{A}$  is a distribution  $\mathcal{E}(x) \in \mathcal{D}'$  that

$$\mathcal{A}\mathcal{E}(x) = \delta(x).$$

The key idea of the fundamental solution is that

$$\mathcal{A}\mathcal{E}(x - y) = \delta(x - y),$$

then (why?)

$$\mathcal{A} \left( \int_{-\infty}^{\infty} \mathcal{E}(x - y) f(y) dy \right) = \int_{-\infty}^{\infty} \delta(x - y) f(y) dy = f(x).$$

It means  $\mathcal{E} * f$  is a solution to the ODE  $\mathcal{A}u = f$ .

**Example 4.3.2.** Let  $\mathcal{A} = \frac{d}{dx}$ , the fundamental solution is the Heaviside function  $H(x)$ . We should notice that the fundamental solution is not necessarily unique.

**Example 4.3.3.** Let  $\mathcal{A} = \frac{d^2}{dx^2}$ , the fundamental solution is  $\frac{1}{2}|x|$ .

**Theorem 4.3.4.** Let  $\mathcal{A}$  be an ordinary differential operator with constant coefficients,

$$\mathcal{A} = \sum_{k=0}^n a_k \frac{d^k}{dx^k}.$$

Then  $u(x) = H(x)\phi(x)$  is a fundamental solution, where

$$\mathcal{A}\phi = 0, \quad \left. \frac{d^k}{dx^k} \phi(x) \right|_{x=0} = 0, \quad 0 \leq k \leq n-2, \quad \left. \frac{d^{n-1}}{dx^{n-1}} \phi(x) \right|_{x=0} = \frac{1}{a_n}.$$

*Proof.* A more direct approach is to use the Laplace transform (see Chapter 7). We can check

$$\frac{d}{dx}(H\phi) = H\phi' + H'\phi = H\phi',$$

since  $\phi(0) = 0$ . This process inductively continues until

$$\frac{d^n}{dx^n}(H\phi) = H\phi^{(n)} + H'\phi^{(n-1)} = H\phi^{(n)} + \delta \frac{1}{a_n}.$$

□

Green's functions can be viewed as a specific fundamental solution subject to some homogeneous boundary conditions.

**Definition 4.3.5.** Let  $\mathcal{A}$  be a linear differential operator. If  $\mathcal{G}$  is the Green's function, then

$$\mathcal{A}\mathcal{G}(x, y) = \delta(x - y), \quad x, y \in \Omega,$$

and  $\mathcal{B}\mathcal{G}(x, y) = 0$  for  $x \in \partial\Omega$ , where  $\mathcal{B}$  corresponds to the boundary condition.

Then the differential equation

$$\mathcal{A}u = f, \quad \mathcal{B}u = 0$$

permits a solution  $u = \int_{\Omega} \mathcal{G}(x, y)f(y)dy$ . This because

$$\mathcal{A}u = \mathcal{A} \int_{\Omega} \mathcal{G}(x, y)f(y)dy = \int_{\Omega} \mathcal{A}\mathcal{G}(x, y)f(y)dy = \int_{\Omega} \delta(x - y)f(y)dy = f(x).$$

The boundary condition of  $u$  is automatically satisfied.

**Example 4.3.6.** Let  $\mathcal{A} = \frac{d^2}{dx^2} - k^2$ ,  $\mathcal{B}$  denotes the Dirichlet boundary condition. The domain  $\Omega = (0, 1)$ . Then, the Green's function satisfies

$$\mathcal{A}\mathcal{G}(x, y) = 0, \quad \{x > y\} \cup \{x < y\}.$$

On either side, the solution can be found

$$\mathcal{G}(x, y) = \begin{cases} A \sinh(kx) + B \cosh(kx), & x < y, \\ C \sinh(k(x - 1)) + D \cosh(k(x - 1)), & x > y. \end{cases}$$

Across the interface  $x = y$ , we need

1. Continuity.  $\mathcal{G}(y^-, y) = \mathcal{G}(y^+, y)$ .
2. Jump.  $\mathcal{G}_x(y^+, y) = \mathcal{G}_x(y^-, y) + 1$ .

At the boundary points, we need  $\mathcal{G}(\pm 1, y) = 0$ . After solving the coefficients,  $B = D = 0$ , and

$$\begin{aligned} A \sinh(ky) &= C \sinh(k(y - 1)), \\ A \cosh(ky) &= C \cosh(k(y - 1)) - \frac{1}{k}. \end{aligned}$$

Thus,

$$A = \frac{\sinh(k(y - 1))}{k \sinh(k)}, \quad C = \frac{\sinh(ky)}{k \sinh(k)}.$$

The Green's function is

$$\mathcal{G}(x, y) = \begin{cases} \frac{\sinh(k(y-1)) \sinh(kx)}{k \sinh(k)}, & x < y, \\ \frac{\sinh(ky) \sinh(k(x-1))}{k \sinh(k)}, & x > y. \end{cases}$$

**Remark 4.3.7.** For a general  $n$ -th order ODE with

$$\mathcal{A} = \sum_{k=0}^n a_k(x) \frac{d^k}{dx^k}.$$

The conditions become

1. Continuity.  $\partial_x^k \mathcal{G}(y^-, y) = \partial_x^k \mathcal{G}(y^+, y), k = 0, 1, \dots, n-2.$
2. Jump.  $\partial_x^{n-1} \mathcal{G}(y^-, y) = \partial_x^{n-1} \mathcal{G}(y^+, y) + \frac{1}{a_n(y)}.$
3. Boundary.  $\mathcal{B}\mathcal{G}(x, y) = 0$  for  $x \in \partial\Omega.$

On either side of the interface  $x = y$ , the solution has  $n$  coefficients undetermined (total  $2n$ ). The continuity eliminates  $n-1$  of them, and the jump condition eliminates one unknown. The homogeneous boundary conditions eliminate  $n$  unknowns.

For a general 2nd-order ODE for which  $a_2 \neq 0$ , let  $u_1(x)$  and  $u_2(x)$  be two solutions that satisfy homogeneous boundary conditions on both sides. For instance, if the domain is  $(a, b)$ , then

$$\mathcal{B}u_1(a) = \mathcal{B}u_2(b) = 0.$$

Then we can write the Green's function as

$$\mathcal{G}(x, y) = \begin{cases} Au_1(x)u_2(y), & x < y, \\ Bu_1(y)u_2(x), & x > y. \end{cases}$$

The continuity condition becomes  $A = B$ . The jump condition becomes

$$Au_1'(y)u_2(y) + \frac{1}{a_n(y)} = Bu_1(y)u_2'(y).$$

Thus,  $A = B = (a_2(y)W[u_1(y), u_2(y)])^{-1}.$

## 4.4 Introduction to Sobolev Spaces

Sobolev spaces are essential in differential equations, where derivatives are understood in the distributional sense. To make the argument more general, we consider  $\Omega \subset \mathbb{R}^d$  as the open domain.

**Definition 4.4.1.** The smooth functions with compact support are denoted by

$$C_0^\infty(\Omega) := \{f \in C^\infty(\Omega) \mid \text{supp } f \subset V \subset\subset \Omega\}.$$

We use  $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$ , which is called the space of test functions.

**Definition 4.4.2.** If  $f\chi_K \in L^1$  for any compact set  $K \subset\subset \Omega$ , then  $f$  is locally integrable, and denoted by  $f \in L_{\text{loc}}^1(\Omega)$ . Similarly, we can define  $L_{\text{loc}}^p(\Omega)$ .

**Example 4.4.3.**  $f(x) = 1/x$  is locally integrable on  $(0, 1)$  (actually, locally integrable on  $\mathbb{R} - \{0\}$ ). It is not globally integrable.

**Definition 4.4.4.** Let  $u \in L_{\text{loc}}^1(\Omega)$ . Then  $v^\alpha \in L_{\text{loc}}^1(\Omega)$  is called the  $\alpha$ -th **weak derivative** of  $u$  if

$$\int_{\Omega} u(x) D^\alpha \phi(x) dx = (-1)^{|\alpha|} \int_{\Omega} v^\alpha(x) \phi(x) dx, \quad \forall \phi \in \mathcal{D}(\Omega),$$

where  $\alpha = (\alpha_1, \dots, \alpha_n)$  is a multi-index and  $D^\alpha = \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}$ .

**Remark 4.4.5.** If the weak derivative  $v^\alpha$  exists, it is unique. If not, then

$$\int_{\Omega} (v_1^\alpha - v_2^\alpha) \phi dx = 0.$$

**Example 4.4.6.** The Dirac function  $\delta(x - x')$  is not in  $L_{\text{loc}}^1(\Omega)$ . Therefore, the Heaviside function or any discontinuous function cannot have 1st-order weak derivative. The function  $\sigma(x) = \max(x, 0)$  has a 1st-order weak derivative as the Heaviside function.

**Lemma 4.4.7.** If  $f \in L^p(\Omega)$ , then  $f \in L_{\text{loc}}^1(\Omega)$ . In other words,  $L^p(\Omega) \subset$

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$$L^1_{\text{loc}}(\Omega).$$

*Proof.* Let  $K \subset\subset \Omega$  be any compact set. Then  $|K| < \infty$  (finite covering), then

$$\int_K |f(x)| dx \leq \left( \int_K |f(x)|^p dx \right)^{1/p} \left( \int_K 1^q dx \right)^{1/q} \leq \|f\|_{L^p} |K|^{1/q} < \infty.$$

□

Next, we provide a characterization of functions in  $L^p_{\text{loc}}(\Omega)$  as the limit of smooth functions. The standard approach to constructing the limiting sequence is to use mollifiers.

**Definition 4.4.8 (Mollification).** Let  $\eta \in C^\infty(\mathbb{R}^d)$  be a bump function supported on  $B_0(1)$  such that  $\|\eta\|_{L^1(\mathbb{R}^d)} = 1$ . Suppose  $u \in L^1_{\text{loc}}(\Omega)$ , the mollified function of  $u$  is

$$u_\varepsilon = \eta_\varepsilon * u, \quad x \in \Omega_\varepsilon := \{x \in \Omega \mid \text{dist}(x, \partial\Omega) > \varepsilon\},$$

where  $\eta_\varepsilon$  is the mollifier:

$$\eta_\varepsilon(x) = \frac{1}{\varepsilon^d} \eta\left(\frac{x}{\varepsilon}\right).$$

**Theorem 4.4.9 (Mollification in  $L^p$ ).** The following statements are true.

1. Suppose  $u \in C(\Omega)$ , then  $u_\varepsilon \rightarrow u$  uniformly on any compact subset of  $\Omega$  as  $\varepsilon \rightarrow 0$ .
2. Suppose  $u \in L^p_{\text{loc}}(\Omega)$ , then  $u_\varepsilon \rightarrow u$  in  $L^p_{\text{loc}}(\Omega)$  as  $\varepsilon \rightarrow 0$ .

*Proof.* For the first statement,

$$\begin{aligned} \sup_{x \in K} |u_\varepsilon(x) - u(x)| &= \sup_{x \in K} \left| \int_{B_x(\varepsilon)} \eta_\varepsilon(x-y)(u(y) - u(x)) dy \right| \\ &\leq \sup_{x \in K, y \in B_x(\varepsilon)} |u(y) - u(x)|. \end{aligned}$$

Since  $u \in C(\Omega)$ , it is uniformly continuous on a compact subset  $K$ . Thus, the RHS converges to zero as  $\varepsilon \rightarrow 0$ .

For the second statement, given a compact subset  $K \subset\subset \Omega$  and  $\delta > 0$ , there exists another compact subset  $K' \supset\supset K$  and  $v \in C(K')$  such that  $\|u - v\|_{L^p(K')} \leq \delta$ .

$\delta$ . Therefore,

$$\begin{aligned} \|u_\varepsilon - u\|_{L^p(K)} &\leq \|u_\varepsilon - v_\varepsilon\|_{L^p(K)} + \|v_\varepsilon - v\|_{L^p(K)} + \|v - u\|_{L^p(K)} \\ &\leq \|u_\varepsilon - v_\varepsilon\|_{L^p(K)} + \|v_\varepsilon - v\|_{L^p(K)} + \delta. \end{aligned}$$

Select  $\varepsilon$  small enough,  $\bigcup_{x \in K} B_x(\varepsilon) \subset K'$  and  $\|v_\varepsilon - v\|_{L^p(K)} < \delta$  since  $v_\varepsilon \rightarrow v$  pointwise uniformly. The first term satisfies

$$\begin{aligned} \|u_\varepsilon - v_\varepsilon\|_{L^p(K)} &= \|\eta_\varepsilon * (u - v)\|_{L^p(K)} \\ &= \left( \int_K \left| \int_{B_x(\varepsilon)} \eta_\varepsilon(x-y)(u(y) - v(y)) dy \right|^p dx \right)^{1/p} \\ &\leq \left( \int_K \left( \int_{B_x(\varepsilon)} \eta_\varepsilon(x-y) dy \right)^{p/p'} \left( \int_{B_x(\varepsilon)} \eta_\varepsilon(x-y) |u(y) - v(y)|^p dy \right) dx \right)^{1/p} \\ &= \left( \int_K \left( \int_{B_x(\varepsilon)} \eta_\varepsilon(x-y) |u(y) - v(y)|^p dy \right) dx \right)^{1/p} \\ &\leq \|u - v\|_{L^p(K')} \leq \delta. \end{aligned}$$

□

**Definition 4.4.10 (Sobolev Spaces,  $W^{k,p}(\Omega)$ ).** Let  $p \in [1, \infty]$ . The Sobolev space

$$W^{k,p}(\Omega) := \{u \in L^1_{\text{loc}}(\Omega) \mid D^\alpha u \in L^p(\Omega), \quad \forall |\alpha| \leq k\}.$$

The Sobolev norm

$$\|u\|_{W^{k,p}(\Omega)} = \left( \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^p(\Omega)}^p \right)^{1/p}, \quad p \in [1, \infty).$$

and

$$\|u\|_{W^{k,\infty}(\Omega)} = \max_{|\alpha| \leq k} \|D^\alpha u\|_{L^\infty(\Omega)}.$$

**Example 4.4.11.** Suppose the function  $f(x) = |x|^{-\alpha} \in W^{1,p}(\Omega)$ , where  $\Omega = B_0(1) \subset \mathbb{R}^d$ . We can find that

1.  $f \in L^1_{\text{loc}}(\Omega)$  means  $\int_\Omega |x|^{-\alpha} dx < \infty$ . Thus  $\alpha < d$ .
2.  $f \in L^p(\Omega)$  means  $p\alpha < d$ .
3.  $D_{x_i} f = -\alpha |x|^{-\alpha-2} x_i \in L^p(\Omega)$  means  $\alpha < \frac{d}{p} - 1$ .

We notice that if  $d < p$ , then  $\alpha < 0$ , which means  $f \in C(\Omega)$ . This verifies Morrey's inequality. We will see that in the Sobolev embedding theorems.

**Theorem 4.4.12.**  $W^{k,p}(\Omega)$  is a Banach space.

*Proof.* Let  $u_k$  be a Cauchy sequence in  $W^{k,p}(\Omega)$ . Therefore, each weak derivative  $D^\alpha u_k$  is a Cauchy sequence in  $L^p(\Omega)$ . Since  $L^p(\Omega)$  is a Banach space.

$$D^\alpha u_k \rightarrow u^\alpha \in L^p(\Omega).$$

Then, we simply show  $u^\alpha$  is the  $\alpha$ -th weak derivative of  $u^0$ , which is the limit of  $u_k$  in the  $L^p$  sense.

Let  $\phi \in C_0^\infty(\Omega)$ , then

$$\begin{aligned} \int_{\Omega} u^0 D^\alpha \phi dx &= \lim_{k \rightarrow \infty} \int_{\Omega} u_k D^\alpha \phi dx \\ &= (-1)^{|\alpha|} \lim_{k \rightarrow \infty} \int_{\Omega} D^\alpha u_k \phi dx = (-1)^{|\alpha|} \int_{\Omega} u^\alpha \phi dx. \end{aligned}$$

Therefore,  $u^\alpha = D^\alpha u^0$ . Thus,  $u_k \rightarrow u^0$  in  $W^{k,p}(\Omega)$ .  $\square$

**Corollary 4.4.13.**  $H^k(\Omega) := W^{k,2}(\Omega)$  is a Hilbert space with inner product

$$\langle u, v \rangle_{H^k} = \sum_{|\alpha| \leq k} \langle D^\alpha u, D^\alpha v \rangle_{L^2}.$$

#### 4.4.1 Approximation By Smooth Functions

The following theorem shows that any function in the Sobolev space can be approximated by smooth functions in the interior domain.

**Theorem 4.4.14 (Local Approximation).** Let  $p \in [1, \infty)$  and  $k \geq 0$ . Suppose  $u \in W^{k,p}(\Omega)$ . Define

$$u_\varepsilon = \eta_\varepsilon * u,$$

where  $\eta_\varepsilon(x) = \frac{1}{\varepsilon^d} \eta\left(\frac{x}{\varepsilon}\right) \in C_0^\infty(B_0(\varepsilon))$ . Then each  $u_\varepsilon \in C^\infty(\Omega_\varepsilon)$  and

$$\lim_{\varepsilon \rightarrow 0} \|u - u_\varepsilon\|_{W^{k,p}(K)} = 0,$$

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for any compact  $K \subset\subset \Omega$ .

*Proof.* It is simple to show  $u_\varepsilon \in C^\infty(\Omega_\varepsilon)$  by definition and  $u \in L^1_{\text{loc}}(\Omega)$ .

For any compact  $K \subset\subset \Omega$ , we choose  $\varepsilon > 0$  that  $K \subset \Omega_\varepsilon$ . Let  $\phi \in C_0^\infty(K)$ , then

$$\begin{aligned} \int_K D^\alpha u_\varepsilon \phi dx &= \int_K \phi(x) \left( \int_\Omega D_x^\alpha \eta_\varepsilon(x-y) u(y) dy \right) dx \\ &= \int_K \left( \int_\Omega \eta_\varepsilon(x-y) D^\alpha u(y) dy \right) \phi(x) dx \\ &= \int_K (\eta_\varepsilon * D^\alpha u) \phi dx. \end{aligned}$$

Thus,  $D^\alpha u_\varepsilon = \eta_\varepsilon * D^\alpha u$  and converges in  $L^p_{\text{loc}}(\Omega)$  as  $\varepsilon \rightarrow 0^+$ .  $\square$

Moreover, the above theorem can be extended from the interior to the boundary, which is known as the *Meyers-Serrin Theorem*.

**Theorem 4.4.15 (Meyers-Serrin Theorem).** Let  $p \in [1, \infty)$ ,  $k \geq 0$ , and  $\Omega$  is bounded. Let  $u \in W^{k,p}(\Omega)$ . Then there exists a sequence  $u_k \in C^\infty(\Omega) \cap W^{k,p}(\Omega)$  such that

$$\lim_{k \rightarrow \infty} \|u_k - u\|_{W^{k,p}(\Omega)} = 0.$$

*Rough idea.* Choosing a sequence of layer open sets  $U_k = \{x \in \Omega \mid \text{dist}(x, \partial\Omega) \in (\frac{1}{k+2}, \frac{1}{k})\}$ . For any  $f \in W^{k,p}(\Omega)$ , decompose the function through a partition of unity:

$$f = \sum_{k=1}^{\infty} \chi_{U_k} f.$$

Then, use the previous local approximation theorem on each  $U_k$ , there is a  $\phi_k \in C^\infty(U_k)$  with an approximation error within  $\frac{\varepsilon}{2^k}$  in  $W^{k,p}$  norm. Then, let  $\phi = \sum_{k=1}^{\infty} \phi_k \in C^\infty(\Omega)$  is an approximation to  $f$  within an error of  $\varepsilon$ .  $\square$

**Remark 4.4.16.**  $W^{k,\infty}(\Omega)$  does not allow smooth approximations in  $W^{k,\infty}$  norm. For instance, the absolute value function is in  $W^{1,\infty}((-1, 1))$ , its derivative is bounded but discontinuous.

**Remark 4.4.17.** Historically, there have been two definitions of Sobolev spaces [Adams and Fournier \(2003\)](#).  $W^{k,p}$  is the set of functions with weak derivatives up to order  $k$  in  $L^p$ . Another definition,  $H^{k,p}$ , is the closure of smooth functions under the  $W^{k,p}$  norm. Meyers-Serrin in 1964 showed that these two definitions are equivalent without any regularity requirement about  $\Omega$ .

**Remark 4.4.18.** If we need the smooth functions in  $C^\infty(\overline{\Omega})$  instead of  $C^\infty(\Omega)$  to approximate  $W^{k,p}(\Omega)$ , that is, bounded derivatives of all orders, then  $\partial\Omega$  needs some regularity. See (Evans, 2022, Section 5.3.3).

#### 4.4.2 Sobolev Embedding

**Definition 4.4.19.** A Banach space  $X$  is **compactly embedded** in another Banach space  $Y$  if  $\|f\|_X \geq C\|f\|_Y$  for some constant  $C = C(X, Y)$  and any  $f \in X$ . It is denoted as  $X \hookrightarrow Y$ .

In this section, we will establish some embedding results for Sobolev spaces, that is, if  $X = W^{k,p}(\Omega)$  and  $Y = W^{m,q}(\Omega)$ , under what kind of situation(s) we can find that  $\|f\|_{W^{k,p}} \geq C\|f\|_{W^{m,q}}$ ?

It is known that  $f \in W^{k,p}(\Omega)$  can be approximated by smooth functions. Suppose we can prove the embedding inequality for smooth functions in  $W^{k,p}(\Omega) \cap C^\infty(\Omega)$ . Does that imply  $f$  also satisfies the same inequality?

**Lemma 4.4.20.** Suppose

$$\|h\|_{W^{k,p}(\Omega)} \geq C\|h\|_{W^{m,q}(\Omega)}$$

for all  $h \in W^{k,p}(\Omega) \cap C^\infty(\Omega)$ . Then for any  $f \in W^{k,p}(\Omega)$ ,

$$\|f\|_{W^{k,p}(\Omega)} \geq C\|f\|_{W^{m,q}(\Omega)}.$$

*Proof.* First, we show that if  $f \in W^{k,p}(\Omega)$ , then  $f \in W^{m,q}(\Omega)$ . Choose a Cauchy sequence  $f_j \in W^{k,p}(\Omega) \cap C^\infty(\Omega)$ . For any  $\varepsilon > 0$ , there exists  $N$  that  $j, l > N$  and

$$\|f_j - f_l\|_{W^{k,p}(\Omega)} < \varepsilon.$$

By the embedding inequality, it means  $f_j$  is a Cauchy sequence in  $W^{m,q}(\Omega)$ . Thus,  $g = \lim_{j \rightarrow \infty} f_j \in W^{m,q}(\Omega)$ . We know that  $\|f_j - f\|_{L^p} \rightarrow 0$  means there exists a subsequence  $f_{j_n} \rightarrow f$  almost everywhere. Since  $g = \lim_{n \rightarrow \infty} f_{j_n}$  in  $W^{m,q}(\Omega)$ , there is a subsequence  $f_{j_{n,s}} \rightarrow g$  almost everywhere. Thus  $f = g$ . The result is immediately proved by noticing that

$$\|f_j\|_{W^{k,p}(\Omega)} \rightarrow \|f\|_{W^{k,p}(\Omega)}, \quad \|f_j\|_{W^{m,q}(\Omega)} \rightarrow \|f\|_{W^{m,q}(\Omega)}.$$

□

We first introduce the Gagliardo-Nirenberg-Sobolev inequality. From the pre-

vious theorem, we only need to show that the inequality holds for smooth functions.

**Theorem 4.4.21 (Gagliardo-Nirenberg-Sobolev).** *Let  $u \in C_0^\infty(\mathbb{R}^d)$  that  $1 \leq p < d$ . Then*

$$\|u\|_{L^{p^*}(\mathbb{R}^d)} \leq C \|Du\|_{L^p(\mathbb{R}^d)}.$$

*The constant only depends on  $p$  and  $d$  and  $p^* = \frac{dp}{d-p} > p$ .*

Therefore,  $W^{1,p}(\mathbb{R}^d) \hookrightarrow L^{p^*}(\mathbb{R}^d)$ . We obtain the famous Poincaré inequality (where  $C$  depends on  $\Omega$ )

$$\|u\|_{L^2(\Omega)} \leq C \|Du\|_{L^2(\Omega)}, \quad \forall u \in W_0^{1,2}(\Omega).$$

The space  $W_0^{1,2}(\Omega) \subset W^{1,2}(\Omega)$  is the closure of  $C_0^\infty(\Omega)$  under  $W^{1,2}(\Omega)$  norm. If  $d < p$ , the space  $W^{1,p}$  has more regularity and can embed into Hölder continuous function space.

**Theorem 4.4.22 (Morrey).** *TODO.*

**Example 4.4.23.** *TODO.*

**Remark 4.4.24.** *For the critical case  $p = d$ , we find  $p^* = \infty$  in the Gagliardo-Nirenberg-Sobolev inequality. However, the following example shows*

$$f(x) = \log(\log(1 + \frac{1}{|x|})) \notin L^\infty(B_0(1)).$$

*But the singularity is very mild, thus*

$$f(x) \in W^{1,p}(B_0(1)), \quad p > 1.$$

### 4.4.3 Application: Sturm-Liouville Problem

Suppose  $\Omega = (a, b) \subset \mathbb{R}$ . Let  $p, q \in L^\infty(\Omega)$  and  $f \in L^2(\Omega)$ . The *regular* Sturm-Liouville equation is

$$-\frac{d}{dx} \left( p(x) \frac{d}{dx} u \right) + q(x)u(x) = f(x), \quad (4.1)$$

where  $p(x), q(x) > \alpha > 0$  for  $x \in \Omega$ , and we consider the homogeneous Neumann boundary conditions  $u'|_{\partial\Omega} = 0$ . The equation can be easily extended to higher

dimensional domain with a smooth boundary.

**Definition 4.4.25.** A weak solution to (4.1) is a function  $u \in W^{1,2}(\Omega)$  that

$$\int_{\Omega} p(x)u'(x)\phi'(x)dx + \int_{\Omega} q(x)u(x)\phi(x)dx = \int_{\Omega} f(x)\phi(x),$$

for all  $\phi(x) \in C^{\infty}(\Omega) \cap W^{1,2}(\Omega)$  which is dense in  $W^{1,2}(\Omega)$ .

In the following, we show that a weak solution  $u \in W^{1,2}(\Omega)$  exists and is unique under certain circumstances. Let

$$B[u, v] = \int_{\Omega} p(x)u'(x)v'(x)dx + \int_{\Omega} q(x)u(x)v(x)dx,$$

The operator  $B[\cdot, \cdot] : W^{1,2}(\Omega) \times W^{1,2}(\Omega) \rightarrow \mathbb{R}$  is bilinear on the Hilbert space  $W^{1,2}(\Omega)$ . We first prove a simple version of the Riesz Representation Theorem, which relies on the following lemma.

**Lemma 4.4.26.**  $\mathcal{H}$  is a separable Hilbert space if and only if it has a countable set of orthogonal bases.

*Proof.* If  $\mathcal{H}$  has a countable set of orthogonal bases  $\{\phi_i\}_{i \in \mathbb{N}}$ , then

$$\left\{ \sum_{i \in \mathbb{N}} c_i \phi_i \mid c_i \in \mathbb{Q} \right\}$$

is a countable dense subset of  $\mathcal{H}$ .

If  $\mathcal{H}$  has an uncountable set of (normalized) orthogonal bases (Zorn's lemma guarantees the existence). Then any dense subset of  $\mathcal{H}$  must be uncountable due to the  $\mathcal{O}(1)$  distance between the orthogonal bases.  $\square$

**Theorem 4.4.27 (Riesz, separable).** Let  $\mathcal{A} : \mathcal{H} \rightarrow \mathbb{R}$  be a bounded linear functional on a **separable** Hilbert space  $\mathcal{H}$ . Then there exists a unique  $f \in \mathcal{H}$  that

$$\mathcal{A}h = \langle f, h \rangle,$$

where  $\langle \cdot, \cdot \rangle$  is the inner product defined on  $\mathcal{H}$ .

*Proof.* Let  $\{\phi_i\}_{i \in \mathbb{N}}$  be a set of normalized orthogonal bases that spans  $\mathcal{H}$ . Then  $\mathcal{A}$  is uniquely determined by  $c_i := \mathcal{A}\phi_i$ . Therefore, we have a unique  $f = \sum_{i=1}^{\infty} c_i \phi_i$ . Since  $\langle f, f \rangle = \sum_{i=1}^{\infty} |c_i|^2$  and

$$\|\mathcal{A}\|_{op}^2 = \sup_{\|h\|=1} |\mathcal{A}h|^2 = \sup_{\sum_{i \in \mathbb{N}} |t_i|^2 = 1} \left( \sum_{i=1}^{\infty} |c_i t_i| \right)^2 = \sum_{i=1}^{\infty} |c_i|^2.$$

We know  $f \in \mathcal{H}$ . □

**Theorem 4.4.28 (Lax-Milgram, separable).** Let  $\mathcal{H}$  be a *separable* Hilbert space. Suppose the bilinear form  $B[\cdot, \cdot] : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$  satisfies

1. *continuous*:  $|B[u, v]| \leq C\|u\|_{\mathcal{H}}\|v\|_{\mathcal{H}}$ ;
2. *coercive*:  $B[u, u] \geq c\|u\|_{\mathcal{H}}^2$ ;

Then the equation  $B[u, v] = \langle f, v \rangle$  has a unique solution  $u \in \mathcal{H}$ .

*Proof.* Since  $|B[u, v]| \leq C\|u\|_{\mathcal{H}}\|v\|_{\mathcal{H}}$ , it means  $B[u, \cdot] : \mathcal{H} \rightarrow \mathbb{R}$  is a bounded linear functional. Thus, by the previous Riesz Representation Theorem, there exists  $\tilde{u} \in \mathcal{H}$  that

$$B[u, v] = \langle \tilde{u}, v \rangle, \quad \forall v \in \mathcal{H}.$$

The map  $\mathcal{J} : u \rightarrow \tilde{u}$  must be linear since

$$\begin{aligned} \langle \mathcal{J}(a_1 u_1 + a_2 u_2), v \rangle &= B[a_1 u_1 + a_2 u_2, v] \\ &= a_1 B[u_1, v] + a_2 B[u_2, v] = \langle a_1 \mathcal{J}u_1 + a_2 \mathcal{J}u_2, v \rangle. \end{aligned}$$

In other words,  $B[u, v] = \langle \mathcal{J}u, v \rangle$  and by the continuity condition,  $\|\mathcal{J}\|_{op} \leq C$ .

We first notice that  $\mathcal{J}$  is injective by

$$B[u, u] = \langle \mathcal{J}u, u \rangle \geq c\|u\|^2 \Rightarrow \|\mathcal{J}u\| \geq c\|u\|.$$

and it also implies  $\text{Ran}(\mathcal{J})$  is closed (why?). We then show that  $\mathcal{J}$  is surjective. If not,  $\exists h \in \mathcal{H}$  and  $h \neq 0$  that  $h \perp \text{Ran}(\mathcal{J})$ . Thus,

$$c\|h\|_{\mathcal{H}}^2 \leq B[h, h] = \langle \mathcal{J}h, h \rangle = 0.$$

Therefore,  $\mathcal{J}^{-1} : \mathcal{H} \rightarrow \mathcal{H}$  exists and  $\|\mathcal{J}^{-1}\| \leq c^{-1}$ . The solution  $u = \mathcal{J}^{-1}f$  is the unique solution. □

**Remark 4.4.29.** The separability of the Hilbert space is NOT necessary for the Riesz Rep-

resentation Theorem or the Lax-Milgram Theorem.

Now, we shift back to the Sturm-Liouville equation. The weak solution satisfies the equation

$$B[u, v] = \langle f, v \rangle_{L^2}.$$

Notice that the RHS is the usual  $L^2$  inner product, not the  $W^{1,2}$  inner product. However, by the Riesz Representation Theorem ( $W^{k,p}$  is separable if  $1 \leq p < \infty$ ), there exists a unique  $\tilde{f} \in \mathcal{H} := W^{1,2}(\Omega)$  that

$$\langle f, v \rangle_{L^2} = \langle \tilde{f}, v \rangle_{\mathcal{H}}.$$

It remains to check if  $B[u, v]$  satisfies the conditions in the Lax-Milgram theorem.

1. continuous: by Cauchy-Schwartz inequality

$$\begin{aligned} |B[u, v]| &= \left| \int_{\Omega} p(x)u'(x)v'(x)dx + \int_{\Omega} q(x)u(x)v(x)dx \right| \\ &\leq (\|p\|_{L^\infty} + \|q\|_{L^\infty})\|u\|_{W^{1,2}}\|v\|_{W^{1,2}}. \end{aligned}$$

2. coercive:

$$B[u, u] = \int_{\Omega} (p(x)|u'(x)|^2 + q(x)|u(x)|^2)dx > \alpha\|u\|_{W^{1,2}}^2.$$

Therefore, there is a unique weak solution  $u \in W^{1,2}(\Omega)$  to the ODE (4.1).

**Remark 4.4.30.** To recover the existence of a strong solution, we can use the bootstrapping strategy.

Suppose  $q \in C^1(\bar{\Omega})$ ,  $p(x)$ ,  $f(x) \in C(\bar{\Omega})$ . Since  $p(x)u'$  has a weak derivative  $f(x) - q(x)u(x) \in L^2(\Omega)$ , thus  $p(x)u' \in H^1(\Omega)$  and  $u \in H^2(\Omega)$ . The Sobolev embedding implies  $u \in C(\bar{\Omega})$ . Then the weak derivative  $f(x) - q(x)u(x) \in C(\bar{\Omega})$  becomes a strong derivative and  $p(x)u'(x) \in C^1(\bar{\Omega})$ . We obtain  $u \in C^2(\bar{\Omega})$ .

## 4.5 Exercises

☛ **Problem 4.5.1.** Let  $\eta \in L^1(\mathbb{R})$  that  $\|\eta\|_{L^1(\mathbb{R})} = 1$ . Show that

$$\frac{1}{\varepsilon} \left| \eta\left(\frac{x}{\varepsilon}\right) \right| \xrightarrow{w} \delta(x)$$

☛ **Problem 4.5.2.** Show that

$$\frac{1}{2\pi i} \left( \frac{1}{x - i\varepsilon} - \frac{1}{x + i\varepsilon} \right) \xrightarrow{w} \delta(x), \quad \mathbb{R} \ni \varepsilon \rightarrow 0^+.$$

Then prove

$$\frac{1}{\pi} \int_0^\infty \cos(xt) dt = \delta(x).$$

☕ **Problem 4.5.3.** Find a consistent way to define the non-integer derivatives of  $\delta(x)$ .

☕ **Problem 4.5.4.** Suppose  $p(x)u''(x) = 0$  for all  $x \in \mathbb{R}$ , where  $p$  is a general polynomial. Find the distribution solution for  $u$ .

☕ **Problem 4.5.5.** Prove that a distribution  $\ell \in \mathcal{D}'$  satisfies that  $\ell^{(k)} = 0$  if and only if  $\ell$  is a polynomial of degree at most  $k - 1$ .

☕ **Problem 4.5.6.** Show the Sobolev space  $W^{k,2}(\mathbb{R}^d)$  forms a Banach algebra if  $k > \frac{d}{2}$ , that is,

$$\|uv\|_{W^{k,2}(\mathbb{R}^d)} \leq C \|u\|_{W^{k,2}(\mathbb{R}^d)} \|v\|_{W^{k,2}(\mathbb{R}^d)}.$$

## Extended Reading

Adams, R. A. and Fournier, J. J. (2003). *Sobolev spaces*, volume 140. Elsevier.

Evans, L. C. (2022). *Partial differential equations*, volume 19. American mathematical society.

Hörmander, L. (2003). *The analysis of linear partial differential operators I: Distribution Theory and Fourier Analysis*. Springer Berlin, Heidelberg.