Chapter 0

Lightning Complex Analysis

Mathematics is the art of giving the same name to different things.

– Henri Poincaré

THIS chapter is a brief tour of basic tools from complex analysis that will be used later. Unfortunately, you will surely miss those fascinating landmarks, such as the Riemann sphere, conformal mappings, the Riemann mapping theorem, elliptic functions, and Zeta functions, along with their attractive applications. If you wish to dive slightly deeper, Stein and Shakarchi (2010) is a good place for a beginner's expedition.

0.1 Complex Numbers

The set of complex numbers $\mathbb{C}$ is defined in the form of $z = x + iy$ where $i = \sqrt{-1}$ is the imaginary unit and x, y are real numbers, called *real* and *imaginary* parts, respectively. We denote $x = \Re(z)$ and $y = \Im(z)$. The set $\mathbb{C}$ is also called the **complex plane**.

Definition 0.1.1. *Let $z = x + iy$, then $x - iy$ is called the* ***complex conjugate*** *of z, denoted as $\overline{z}$, and $|z| = (x^2 + y^2)^{1/2}$ is the* ***modulus*** *of z.*

It is fairly simple to verify $\Re(z) = \frac{1}{2}(z + \overline{z})$, $\Im(z) = \frac{1}{2}(z - \overline{z})$, $|z|^2 = z\overline{z}$, and the complex conjugate commutes with basic operations, e.g., $\overline{z + w} = \overline{z} + \overline{w}$, $\overline{zw} = \overline{z} \cdot \overline{w}$, and $\overline{z/w} = \overline{z}/\overline{w}$.

Another representation of complex numbers borrows from the polar coordinates in two dimensions. A complex number $z = x + iy = r(\cos\phi + i\sin\phi)$, where $r = |z|$ is the **modulus** and ϕ is called the **argument**. If $z = r(\cos\phi + i\sin\phi)$ and $w = \rho(\cos\psi + i\sin\psi)$, then if $zw = a(\cos\theta + i\sin\theta)$, then

$$\begin{aligned} zw &= r\rho(\cos\phi + i\sin\phi)(\cos\psi + i\sin\psi) \\ &= r\rho(\cos(\phi + \psi) + i\sin(\phi + \psi)), \end{aligned}$$

which implies that we can choose $a = r\rho$ and $\theta = \phi + \psi + 2k\pi$ for any $k \in \mathbb{Z}$. To eliminate such an ambiguity, we define the principal argument of z, denoted by $\arg(z) \in [0, 2\pi)$. The set of all arguments is denoted by

$$\operatorname{Arg} z = \{\arg z + 2k\pi \mid k \in \mathbb{Z}\}.$$

If we regard the principal argument $\arg z$ as a function for $z \in \mathbb{C} - \{0\}$ with a range in $[0, 2\pi)$ or $(-\pi, \pi]$, then we find this function is discontinuous on the half-line $(0, \infty)$ or $(-\infty, 0)$. The latter one is preferred because it ensures continuity on the real axis, especially when dealing with complex logarithms. See the example below.

Example 0.1.2. *A more elegant representation of a complex number $a + bi$ is the matrix form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. For instance, the product and quotient of two complex numbers follow*

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} ac - bd & -(bc + ad) \\ (bc + ad) & ac - bd \end{bmatrix}$$

and

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & -d \\ d & c \end{bmatrix}^{-1} = \frac{1}{c^2+d^2} \begin{bmatrix} ac+bd & ad-bc \\ bc-ad & ac+bd \end{bmatrix}.$$

Example 0.1.3 (De Moivre's formula)**.** *The complex number* $\cos\phi + i\sin\phi$ *is often represented by* $e^{i\phi}$*. Since the exponential function's Taylor expansion converges absolutely*

$$e^{i\phi} = \sum_{k=0}^{\infty} \frac{1}{k!}(i\phi)^k \overset{\text{why?}}{=\!=} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\phi^{2k} + i\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\phi^{2k+1}$$
$$= \cos(\phi) + i\sin(\phi).$$

Example 0.1.4. *The unit disk* $D(0,1)$ *consists of complex numbers that* $|z| < 1$*. We introduce a few common transformations.*

1. *Transformation* $g(z) = \frac{1+z}{1-z}$*.*

   It is a ***Möbius transformation*** *in the form of* $\frac{a+bz}{c+dz}$*, which preserves angles, and maps every straight line to a line or circle, and maps every circle to a line or circle.*

   Indeed, the transformation $\frac{1}{1-z}$ *maps the disk's boundary* $z = e^{i\theta}$ *to a line:*

   $$\frac{1}{1-e^{i\theta}} = \frac{1-\cos\theta + i\sin\theta}{2-2\cos\theta} = \frac{1}{2} + \frac{i}{2}\frac{\sin\theta}{1-\cos\theta} = \{z \mid \Re(z) = \frac{1}{2}\}.$$

   and maps the origin to $z = 1$*. Thus,* $\frac{1}{1-z}$ *bijectively maps the disk to* $\{\Re(z) > \frac{1}{2}\}$*. Therefore, the transform* $g(z) = \frac{1+z}{1-z} = -1 + \frac{2}{1-z}$ *bijectively maps the disk to* $\{\Re(z) > 0\}$*, i.e., the right half-plane.*

2. *Transformation* $h(z) = \log g(z)$*. It is known* $w = g(z)$ *belongs to the right half-plane, we can write* $w = |w|e^{i\psi}$*, that* $\psi \in (-\pi/2, \pi/2)$*. Then*

   $$h(z) = \log w = \log|w| + i\psi = \{z \mid -\frac{\pi}{2} < \Im z < \frac{\pi}{2}\}.$$

   Therefore, h *bijectively maps the disk to a strip.*

0.2 Calculus of Complex Variables

We can extend undergraduate calculus from $\mathbb{R}$ to $\mathbb{C}$. Let $f : \mathbb{C} \to \mathbb{C}$ be a complex-valued function (assume this function is single-valued). Then, we can first define the "limit" as usual.

Definition 0.2.1. *Given any $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(z) - A|$ for all $|z - a| < \delta$ $(z \neq a)$. Then $\lim_{z\to a} f(z) = A$. If $f(a) = A$, then it implies $f(z)$ is continuous at $z = a$.*

0.2.1 Differentiation

Most basic concepts can be extended naturally as if we were on a two-dimensional plane. For "differentiation", it is straightforward to consider the following limit

$$\lim_{|h|\to 0} \frac{f(z+h) - f(z)}{h}.$$

If such a limit exists and is equal for all limiting processes $h \to 0$ on the complex plane, then f is differentiable at z, and the limit is denoted by $f'(z)$.

Definition 0.2.2. *If $f(z)$ is differentiable on $D \subset \mathbb{C}$, then $f(z)$ is called* ***analytic*** *or* ***holomorphic*** *on D.*

Let $f(z) = u(z) + iv(z) = u(x, y) + iv(x, y)$, then f is differentiable at z implies two path limits

$$\lim_{\mathbb{R}\ni h\to 0} \frac{u(x+h, y) + iv(x+h, y) - (u(x, y) + iv(x, y))}{h} = \partial_x u + i\partial_x v$$

and

$$\lim_{\mathbb{R}\ni h\to 0} \frac{u(x, y+h) + iv(x, y+h) - (u(x, y) + iv(x, y))}{ih} = -i\partial_y u + \partial_y v.$$

This means we must have

$$u_x = v_y, \quad u_y = -v_x.$$

This equation is called the **Cauchy-Riemann equation**. Of course, holding this equation at z is only a necessary condition to make f differentiable at z.

Example 0.2.3 (Cauchy-Riemann at a point). *If we take $f(z) = f(x+iy) = \sqrt{|xy|}$, then f satisfies the Cauchy-Riemann equation at $z = 0$ while the differential $f'(z)$ does not exist.*

The following theorem implies that u and v must be differentiable in D, as well.

Theorem 0.2.4. *If $f(z) = u + iv$ satisfies that u, v are continuously differentiable functions in D and satisfy the Cauchy-Riemann equation, then f is holomorphic in D.*

Proof. By checking the definition, let $h = a + bi$,

$$\begin{aligned}
&\lim_{a^2+b^2\to 0} \frac{u(x+a, y+b) + iv(x+a, y+b) - (u(x,y) + iv(x,y))}{a+bi} \\
&= \lim_{a^2+b^2\to 0} \frac{au_x(x,y) + bu_y(x,y) + iav_x(x,y) + ibv_y(x,y) + o(|h|)}{a+bi} \\
&= u_x + iv_x = \frac{1}{2}(u_x + iv_x) - \frac{i}{2}(u_y + iv_y).
\end{aligned}$$

□

Therefore, we introduce two important operators:

$$\partial_z = \frac{1}{2}\left(\partial_x - i\partial_y\right), \quad \partial_{\bar{z}} = \frac{1}{2}\left(\partial_x + i\partial_y\right).$$

These can be derived from the chain rule. With the operator $\partial_{\bar{z}}$, we can define holomorphic functions by $\partial_{\bar{z}} f = 0$ and the complex derivative of f is $\partial_z f$.

Example 0.2.5. *When f and g are holomorphic in D, the differentiation rule of the product fg (and the quotient f/g, resp.) is the same as that of a real variable.*

Remark 0.2.6. *If $\partial_z f$ is also holomorphic (it is true by the Cauchy Integral Formula; for now, we accept that), then u and v are both continuously twice differentiable such that $\partial_x^2 u = \partial_{xy} v = -\partial_y^2 u$. It imples u (v as well) should satisfy the* ***Laplace*** *equation*

$$\Delta u = 0, \quad \Delta := \partial_x^2 + \partial_y^2,$$

and we say u is a ***harmonic*** *function. It is straightforward to verify that*

$$\Delta = 4\partial_z\partial_{\bar{z}} = 4\partial_{\bar{z}}\partial_z.$$

0.2.2 Path Integration

Let γ be a piecewise differentiable arc on $\mathbb{C}$ described by $z = h(t)$, $t \in [\alpha, \beta]$. Then the path integration of a complex valued function $f(z) = u(z) + iv(z)$ on γ can be written as

$$\int_\gamma f(z)dz := \int_\alpha^\beta f(h(t))h'(t)dt.$$

This integral is independent of the arc variable by the chain rule. Similar to the path integrals of real functions, we can define the **reversed** path integral

$$\int_{-\gamma} f(z)dz = -\int_{\gamma} f(z)dz.$$

And if γ consists of a finite number of parts, that is, $\gamma = \sum_{i=1}^{n} \gamma_i$, then

$$\int_{\gamma} f(z)dz = \sum_{i=1}^{n} \int_{\gamma_i} f(z)dz.$$

The path integral for real variables is connected to **Green's theorem** when the path γ is a simple **closed** curve (oriented counterclockwise, see Figure 1). The

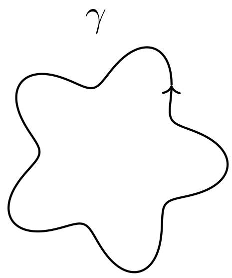

Figure 1: Simple closed path γ.

corresponding version for complex variables can be similarly established. It can be written in a much simpler form with **exterior calculus**, see Gong and Gong (2007). Assuming that f is **continuously differentiable**, then

$$\begin{aligned}
\oint_{\gamma} f(z)dz &= \oint_{\gamma} (u(x,y) + iv(x,y))(dx + idy) \\
&= \oint_{\gamma} u(x,y)dx - v(x,y)dy + i \oint_{\gamma} u(x,y)dy + v(x,y)dx \\
&= \iint_{D} (-\partial_x v - \partial_y u)dxdy + i \iint_{D} (u_x - v_y)dxdy \\
&= i \iint_{D} (\partial_x + i\partial_y) f(z)dxdy = 2i \iint_{D} \partial_{\bar{z}} f(z)dxdy.
\end{aligned} \tag{1}$$

Here, we do not require f to be holomorphic.

Remark 0.2.7. *Once f is holomorphic in D (surrounded by a simple closed path γ), then using the Cauchy-Riemann equation, the path integral of $f(z)$ on γ must equal zero!*

0.2.3 Complex Series

The theory of real series can be generalized to complex series without much effort. The following definitions are standard.

Definition 0.2.8. *A sequence of functions* $\{f_n(z)\}$ ***converges uniformly*** *to* $f(z)$ *on a set* $E \subset \mathbb{C}$ *if for any* $\varepsilon > 0$*, there exists a positive integer* n_0 *such that*

$$|f_n(z) - f(z)| < \varepsilon$$

for all $n \geq n_0$ *and all* $z \in E$*.*

Definition 0.2.9 (Cauchy Sequence). *A sequence of functions* $\{f_n(z)\}$ *converges uniformly on a set* $E \subset \mathbb{C}$ *if and only if for any* $\varepsilon > 0$*, there exists a positive integer* n_0 *such that*

$$|f_m - f_n| < \varepsilon$$

for all $m, n \geq n_0$ *and all* $z \in E$*.*

The following Weierstrass M-test is a universal tool in analysis. It can be easily proved using the Cauchy sequence, so we skip the proof.

Theorem 0.2.10 (Weierstrass M-test). *Suppose that*

$$f_1(z) + f_2(z) + \cdots + f_n(z) + \cdots$$

is a series of functions defined on a set $E \subset \mathbb{C}$*, and* $M_1 + M_2 + M_3 + \cdots$ *is a series of positive numbers. If there exist a positive integer* n_0 *and a constant* $M > 0$ *such that* $f_n(z) \leq M_n$ *for* $n \geq n_0$ *and all* $z \in E$*, then the series* $\sum_{n=1}^{\infty} f_n(z)$ *converges uniformly on* E *if the series* $\sum_{n=1}^{\infty} M_n$ *converges.*

The power series is the most common in literature. The following **Abel's theorem** is more or less the same as the real-valued one.

Theorem 0.2.11 (Abel). *For the power series* $\sum_{n=0}^{\infty} a_n z^n$*, there exists a number* R*,* $0 \leq R \leq \infty$*, called the* ***radius of convergence****, with the following properties:*

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1. *For any z satisfying $|z| < R$, the series is absolutely convergent. The series is uniformly convergent on $|z| \leq \rho$, where $0 \leq \rho < R$.*

2. *If z satisfies $|z| > R$, then the terms of the series are unbounded and the series is divergent.*

3. *The sum of the series is a holomorphic function in the disk $|z| < R$. The derivative of the sum can be obtained by term-wise differentiation, and the resulting series has the same radius of convergence.*

The closed disk $|z| \leq R$ is called the **disk of convergence**. The series is not necessarily convergent or divergent on the boundary of this disk. The radius of convergence R can be determined by *Hadamard's formula*

$$R = \frac{1}{\lim_{n\to\infty} \sup \sqrt[n]{|a_n|}}$$

If $R = 0$, then the power series is divergent except at $z = 0$; If $R = \infty$, then the power series is convergent everywhere.

Proof. (i). If $|z| < R$, we can find a ρ with $|z| < \rho < R$. It follows that $1/\rho > 1/R$. By definition of the radius, there exists an n_0 such that $|a_n|^{\frac{1}{n}} < 1/\rho$ or $|a_n| < 1/\rho^n$ for all $n \geq n_0$. Therefore, $|a_n z^n| < (|z|/\rho)^n$ for $n \geq n_0$. Since $\sum_{n=0}^{\infty}(|z|/\rho)^n$ is convergent for $|z| < \rho$, by the Weierstrass M-test, $\sum_{n=0}^{\infty} a_n z^n$ is absolutely convergent.

To show that the series is uniformly convergent when $|z| \leq \rho < R$, we choose a ρ' and an n_1 such that $\rho < \rho' < R$ and $|a_n z^n| \leq (\rho/\rho')^n$, for $n \geq n_1$. The power series is uniformly convergent on $|z| \leq \rho$ by the Weierstrass M-test.

(ii). If $|z| > R$, we can find a ρ with $R < \rho < |z|$. Since $1/\rho < 1/R$, there exists an n_2 and a subsequence $\{m_i\}$ such that $|a_{m_i}|^{1/m_i} > 1/\rho$ or $|a_{m_i}| > 1/\rho^{m_i}$ for $m_i \geq n_2$. Therefore, $|a_n z^n| > (|z|/\rho)^n$ for infinitely many n, and the terms of the series are unbounded.

(iii). Noticing that the radius of convergence does not change for the derivative series, because $\lim_{n\to\infty} n^{1/n} = 1$. □

0.3 Contour Integrals

The theory of contour integral is built upon the following *Cauchy Integral Formula*. However, the original form developed by Cauchy *assumes* that the derivative of f is continuous (Green's theorem holds). However, the Cauchy-Riemann equation does not imply the *continuity*, only the existence!

0.3.1 Cauchy's Integral

Cauchy's integral is one of the most powerful tools in complex analysis. We first introduce a definition about **connectedness**, which will be used later.

Definition 0.3.1. *An open set U is "connected" if it cannot be written as a disjoint union of two open sets $U = U_1 \cup U_2$.*

A direct consequence of connectedness is that if a subset $S \subset U$ is open and closed, then either $S = \emptyset$ or $S = U$.

Theorem 0.3.2 (Cauchy Integral Formula). *Let $U \subset \mathbb{C}$ be a bounded open **connected** set with C^1 boundary. Let f be holomorphic on U and $f \in C^1(\overline{U})$, then*

$$f(\zeta) = \frac{1}{2\pi i} \oint_{\partial U} \frac{f(z)}{z-\zeta} dz$$

Proof. Let $D_{\zeta,\varepsilon} \subset U$ be a small disc centered at z with a radius of ε. The function $\frac{f(z)}{z-\zeta}$ is holomorphic on $U \backslash D_{\zeta,\varepsilon}$, then by the Green's theorem from (1),

$$\oint_{\partial U} \frac{f(z)}{z-\zeta} dz - \oint_{\partial D(\zeta,\varepsilon)} \frac{f(z)}{z-\zeta} dz = 0.$$

The above equality does not depend on ε. On the small circle $\partial D_{\zeta,\varepsilon}$, the path integral is

$$\oint_{\partial D_{\zeta,\varepsilon}} \frac{f(z)}{z-\zeta} dz = \int_0^{2\pi} \frac{f(\zeta+\varepsilon e^{i\theta})}{\varepsilon e^{i\theta}} i\varepsilon e^{i\theta} d\theta = i \int_0^{2\pi} f(\zeta+\varepsilon e^{i\theta}) d\theta \xrightarrow{\varepsilon\to 0} 2\pi i f(\zeta).$$

□

By taking $f(z)$ as $(z-\zeta)f(z)$ into the Cauchy Integral Formula (f is holomorphic), we immediately recover Green's theorem.

Theorem 0.3.3 (Cauchy Integral Theorem). *Let $U \subset \mathbb{C}$ be a bounded open connected set with C^1 boundary. If $f \in C^1(\overline{U})$ is holomorphic, then*

$$\oint_{\partial U} f(z) dz = 0.$$

When f is not holomorphic, the Cauchy integral formula becomes Pompieu's formula, which has an additional term from (1),

$$f(\zeta) = \frac{1}{2\pi i}\oint_{\partial U}\frac{f(z)}{z-\zeta}dz - \frac{1}{\pi}\int_{U}\frac{\partial_{\overline{z}} f(z)}{z-\zeta}dxdy.$$

0.3.2 Cauchy-Goursat Theorem

It was later shown by Goursat (and others) that the condition that $f \in C^1(\overline{U})$ can be replaced by $f \in C(\overline{U})$. For the fascinating history behind the theorem, see Bak and Popvassilev (2017).

Theorem 0.3.4 (Cauchy-Goursat Formula). *Suppose $U \subset \mathbb{C}$ is a bounded, open, connected set and ∂U is a simple closed curve. If f is holomorphic on U and continuous on $\overline{U}$, then*

$$f(\zeta) = \frac{1}{2\pi i}\oint_{\partial U}\frac{f(z)}{z-\zeta}dz.$$

Theorem 0.3.5 (Cauchy-Goursat Theorem). *Suppose $U \subset \mathbb{C}$ is a bounded open connected set and ∂U is a simple closed curve. If f is holomorphic on U and continuous on $\overline{U}$, then*

$$\oint_{\partial U} f(z)dz = 0.$$

The proof relies on the observation that the inscribing polygon can effectively approximate any curve. That is, given any $\varepsilon > 0$, there exists an inscribing polygon Γ of U (see Figure 2).

$$\left|\oint_{\partial U} f(z)dz - \oint_{\partial \Gamma} f(z)dz\right| < \varepsilon.$$

For any polygon (maybe nonconvex), it can be decomposed into disjoint triangles $\triangle_i, i = 1, 2, \cdots, N$, see Figure 3. The integral on $\partial\Gamma$ can be written as

$$\oint_{\partial\Gamma} f(z)dz = \sum_{i=1}^{N}\oint_{\partial\triangle_i} f(z)dz,$$

because the integrals on the interior edges are cancelled. Therefore, we only have to prove that Theorem 0.3.5 holds for a triangle, and then by linearity, the theorem

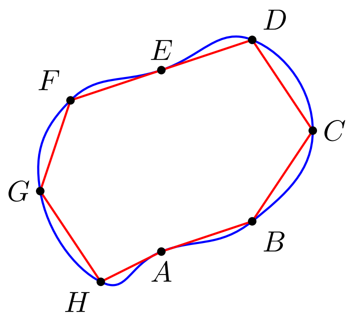

Figure 2: Inscribing polygon path of a closed curve.

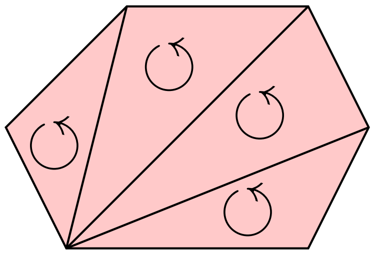

Figure 3: Illustration of triangulation.

holds for any polygon.

Proof. Suppose the Theorem 0.3.5 does not hold for a certain triangle $\triangle^{(0)}$, let

$$M = \left| \oint_{\partial\triangle} f(z)dz \right| > 0.$$

Then we bisect the triangle into four identical smaller triangles $\triangle_1, \triangle_2, \triangle_3, \triangle_4$. Thus

$$\oint_{\partial\triangle} f(z)dz = \left(\oint_{\partial\triangle_1} + \oint_{\partial\triangle_2} + \oint_{\partial\triangle_3} + \oint_{\partial\triangle_4} \right) f(z)dz.$$

There exists at least one smaller triangle $\triangle_k$ (denoted by $\triangle^{(1)}$) that

$$\left| \oint_{\partial\triangle_k} f(z)dz \right| \geq \frac{M}{4}.$$

Repeating this process, we find a chain of triangles: $\triangle^{(0)} \supset \triangle^{(1)} \supset \cdots \supset \triangle^{(n)} \supset \cdots$, and

$$\left| \oint_{\partial\triangle^{(n)}} f(z)dz \right| \geq \frac{M}{4^n}.$$

Then, the circumstance of $\triangle^{(n)}$ is $2^{-n}L$, where $L = |\partial\triangle^{(0)}|$. Taking $n \to \infty$, there exists a point $z_0 \in \triangle^{(n)}$ for all $n \in \mathbb{N}$. Because $f(z)$ is holomorphic (complex-differentiable), there exists $\delta = \delta(\varepsilon)$ that

$$\left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| < \varepsilon, \quad \forall |z - z_0| < \delta.$$

Therefore, for sufficiently large n that $\triangle^{(n)} \subset D(z_0, \delta)$,

$$\begin{aligned}\oint_{\partial\triangle^{(n)}} f(z)dz &= \oint_{\partial\triangle^{(n)}} (f(z) - f(z_0) - (z - z_0)f'(z_0))dz \\ &< \varepsilon \oint_{\triangle^{(n)}} |z - z_0||dz| < \varepsilon \frac{L}{2^n}\frac{L}{2^n} = \varepsilon\frac{L^2}{4^n}.\end{aligned}$$

Here, we have used $|z - z_0| < |\partial\triangle^{(n)}|$ (prove it). Therefore, we conclude $M < \varepsilon L^2$ for any $\varepsilon > 0$, forcing $M = 0$. □

Remark 0.3.6. *A similar idea can be used to prove a simplified version of Sard's theorem. See a proof in (Milnor and Weaver, 1997, Chapter 3, page 16-19).*

0.3.3 Taylor Series

The most critical consequence of Goursat's result is that if f is complex differentiable and continuous, then f is **infinitely** complex differentiable.

Theorem 0.3.7 (Taylor Series). *Suppose $U \subset \mathbb{C}$ is a bounded open connected set and ∂U is a simple closed curve. If f is holomorphic on U and continuous on $\overline{U}$, then*

$$f^{(n)}(\zeta) = \frac{n!}{2\pi i}\oint_{\partial U} \frac{f(z)}{(z - \zeta)^{n+1}}dz.$$

If $z_0 \in U$, then $f(\zeta)$ has the Taylor expansion on the disk $D(z_0, r) \subset U$,

$$f(\zeta) = \sum_{k=0}^{\infty} a_k(\zeta - z_0)^k, \quad a_k = \frac{1}{n!}f^{(k)}(z_0),$$

which converges absolutely and uniformly.

Proof Sketch. We only focus on the induction step $n \to n + 1$. Therefore,

$$\frac{f^{(n)}(\zeta) - f^{(n)}(\zeta_0)}{\zeta - \zeta_0} = \frac{n!}{2\pi i}\oint_{\partial U} f(z)\frac{1}{\zeta - \zeta_0}\left(\frac{1}{(z - \zeta)^{n+1}} - \frac{1}{(z - \zeta_0)^{n+1}}\right)dz.$$

and apply the following estimate (prove it).

$$\frac{1}{\zeta-\zeta_0}\left(\frac{1}{(z-\zeta)^{n+1}}-\frac{1}{(z-\zeta_0)^{n+1}}\right)=\frac{(n+1)}{(z-\zeta_0)^{n+2}}+O(|\zeta-\zeta_0|).$$

Then, taking $\zeta\to\zeta_0$ completes the induction step. The Taylor expansion follows the observation that

$$\frac{1}{z-\zeta}=\frac{1}{z-z_0}\frac{1}{1-\frac{\zeta-z_0}{z-z_0}}=\frac{1}{z-z_0}\sum_{k=0}^{\infty}\left(\frac{\zeta-z_0}{z-z_0}\right)^k$$

converges absolutely. Then apply the Cauchy Integral Formula, then commute integral and summation (why?).

$$\begin{aligned}
f(\zeta)=\frac{1}{2\pi i}\oint_{\partial U}\frac{f(z)}{z-\zeta}dz &= \frac{1}{2\pi i}\oint_{\partial U}\sum_{k=0}^{\infty}\frac{f(z)}{(z-z_0)^{k+1}}(\zeta-z_0)^k dz\\
&=\sum_{k=0}^{\infty}\left(\frac{1}{2\pi i}\oint_{\partial U}\frac{f(z)}{(z-z_0)^{k+1}}dz\right)(\zeta-z_0)^k\\
&=\sum_{k=0}^{\infty}\frac{f^{(k)}(z_0)}{k!}(\zeta-z_0)^k.
\end{aligned}$$

□

Corollary 0.3.8 (Liouville's Theorem). *If $f(z)$ is holomorphic and bounded on the entire $\mathbb{C}$, then f is a constant.*

Proof. Let M be the bound for f and $z_0\in\mathbb{C}$ be an arbitrary complex number. Take a large disk $D(z_0,R)\subset\mathbb{C}$, the derivative

$$f'(z_0)=\frac{1}{2\pi i}\oint_{\partial D(z_0,R)}\frac{f(z)}{(z-z_0)^2}dz.$$

The magnitude is bounded by

$$|f'(z_0)|\le\frac{1}{2\pi}\frac{M}{R^2}\cdot 2\pi R=\frac{M}{R}\xrightarrow{R\to\infty}0.$$

Therefore, $f'(z)\equiv 0$ everywhere. □

Corollary 0.3.9 (Morera Theorem). *If f is continuous on U such that for any simple closed curve C contained in U,*

$$\oint_C f(z)dz = 0,$$

then f is holomorphic on U.

Proof. Choose any $z_0 \in U$ and define $F(z)$ by the path integral along a simple curve connecting z_0 and z

$$F(z) = \int_{z_0 \rightsquigarrow z} f(\zeta)d\zeta.$$

The integral is path independent, and $F'(z) = f(z)$. Therefore, F is holomorphic and by Theorem 0.3.7, all its derivatives are holomorphic. □

Remark 0.3.10. *The requirement of "any simple closed curve" can be further narrowed down to specific shapes. See exercise (☕).*

0.3.4 Zeros of Holomorphic Functions

Let f be a holomorphic function on $U \subset \mathbb{C}$. A point z_0 is called *a zero* or *a root* of f if $f(z_0) = 0$. A quick corollary from the Corollary 0.3.8 (Liouville theorem) shows that

Corollary 0.3.11 (Existence of Zeros). *Let $p(z) = a_0 + a_1 z + \cdots + a_m z^m$ be a nonconsant polynomial. Then $p(z)$ has at least one root.*

Proof. Prove by contradiction. If not, then $\frac{1}{p(z)}$ is holomorphic on $\mathbb{C}$. If the polynomial is not a constant, then $|p(z)| \to \infty$ as $|z| \to \infty$, which makes $\frac{1}{p(z)}$ bounded on $\mathbb{C}$. Then, by Liouville's theorem, $p(z)$ must be a constant. □

When the polynomial $p(z)$ has a root z_0, one can rewrite the polynomial as

$$p(z) = (z - z_0)q(z), \quad q(z) := b_1 + \cdots + b_m(z - z_0)^{m-1},$$

where $q(z)$ is a non-constant polynomial of degree $m - 1$, then an induction argument immediately implies the following theorem.

Theorem 0.3.12 (Fundamental Theorem of Algebra). *A polynomial $p(z)$ of degree m has exactly m roots. If these roots are denoted by $z_1, \cdots, z_m$, then*

$$p(z) = a_m(z - z_1)\cdots(z - z_m).$$

Proof. The proof is left as an exercise (☞). □

In general, the behavior of a holomorphic function around a root can be characterized by the following theorem 0.3.14. Before proving the theorem, we first prove a preparation lemma.

Lemma 0.3.13. *If f is holomorphic in an open connected set $U \subset \mathbb{C}$, and vanishes on a sequence of distinct points with an accumulating point in U. Then $f \equiv 0$ around the accumulating point.*

Proof. Suppose $z_m \to z^*$ in U and $f(z_m) = 0$, then by continuity $f(z^*) = 0$. Inside a small disk $D(z^*, r)$, the Taylor series

$$f(z) = \sum_{k=0}^{\infty} c_k (z - z^*)^k.$$

If f is not identically zero near z^*, then there exists a smallest integer $n > 0$ that $c_n \neq 0$, and

$$f(z) = c_n (z - z^*)^n \left[1 + g(z - z^*)\right],$$

where $g(z - z^*) \to 0$ as $z \to z^*$. Therefore, taking z_n sufficiently close to z^*, we must have $f(z_m) \neq 0$, contradiction! □

Theorem 0.3.14. *If f is holomorphic in an open connected set $U \subset \mathbb{C}$, has a zero at z_0, and does not vanish identically in U. Then there exists a disk $D(z_0, r) \subset U$, a holomorphic function g that $g(z_0) \neq 0$, and a unique positive integer n such that*

$$f(z) = (z - z_0)^n g(z), \quad \forall z \in D(z_0, r).$$

Proof. Around the root z_0, the Taylor series is written as

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

First, we show f is not identically near z_0. Otherwise, consider the **interior** of the set $S = \{z \in U \mid f(z) = 0\}$. It is nonempty and open by definition, and it is closed by the above Lemma 0.3.13. Hence, $S = U$ using connectedness.

Then, there exists a smallest integer $n > 0$ that $a_n \neq 0$, and

$$f(z) = (z - z_0)^n [a_n + a_{n+1}(z - z_0) + \cdots] = (z - z_0)^n g(z).$$

By continuity, $g(z) \neq 0$ in a small neighborhood of z_0, which means the roots are isolated. □

Remark 0.3.15. *A direct consequence of the above theorem is that the roots of a nonzero holomorphic function cannot have accumulating points (but can be infinitely many).*

The integer n is called the multiplicity of the root. Observe that

$$\frac{f'(z)}{f(z)} = \frac{n(z - z_0)^{n-1} g(z) + (z - z_0)^n g'(z)}{(z - z_0)^n g(z)} = \frac{n}{z - z_0} + \frac{g'(z)}{g(z)}$$

Taking a simple closed curve $\gamma \subset U$ that g does not vanish on γ. The function f'/f is holomorphic in the interior of γ after removing the small disks around the roots inside (denoted by $z_1, \cdots, z_m$, with multiplicities $n_1, \cdots, n_m$, respectively). Therefore,

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \sum_{i=1}^{m} \frac{1}{2\pi i} \int_{D(z_i, \varepsilon)} \left(\frac{n_i}{z - z_i} + \frac{g'(z)}{g(z)} \right) dz = \sum_{i=1}^{m} n_i.$$

This is summarized in the following **Argument Principle** without considering poles.

Theorem 0.3.16 (Argument Principle). *Let f be a holomorphic function in an open connected set $U \subset \mathbb{C}$. $\gamma \subset U$ is a simple closed curve. If $f \neq 0$ on γ, then the number of roots (count multiplicity) in the interior of γ is $\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz$.*

Theorem 0.3.17 (Rouché Theorem). *If f and g are holomorphic on $U \subset \mathbb{C}$, and γ is a simple closed curve that lies inside U such that*

$$|f(z) - g(z)| < |f(z)|$$

for all $z \in \gamma$, then f and g have the same number of zeros enclosed by γ.

Proof Sketch. The idea is to show (by the **Argument Principle**),

$$\int_\gamma \left(\frac{g'(z)}{g(z)} - \frac{f'(z)}{f(z)}\right) dz \xlongequal{why?} \int_\gamma \frac{(g/f)'}{(g/f)} dz = 0.$$

Because $|g/f - 1| < 1$ is given, it means the function g/f is not winding around the origin or passing through, and the equality holds by the **Cauchy Integral Formula**. □

Example 0.3.18. *For any polynomial* $p_n(z) = a_0 + a_1 z + \cdots a_n z^n$, $a_n \neq 0$, *we can choose* $q(z) = a_n z^n$, *then*

$$|p_n(z) - q(z)| = |a_0 + \cdots + a_{n-1} z^{n-1}| < |a_n z^n|$$

when $|z|$ *is sufficiently large. Therefore, the polynomial* p_n *has the same number of roots (multiplicity counted) as* $a_n z^n$, *which is* n.

Remark 0.3.19. *For readers with a background in probability, the study of zeros of a random complex polynomial is a longstanding topic. For instance, the expected number of real roots of a random polynomial of degree* n *is roughly* $\frac{2}{\pi} \log n$, *see Edelman and Kostlan (1995) and the earlier references therein.*

0.3.5 Maximum Modulus Principle & Schwartz Lemma

Another important consequence of the *Cauchy Integral Formula* is the *Maximum Modulus Principle*, which is a useful tool.

> **Theorem 0.3.20** (Maximum Modulus Principle)**.** *If* f *is holomorphic in an open connected set* $U \subset C$ *and there exists* $z_0 \in U$ *that* $|f(z_0)| \geq |f(z)|$ *for all* $z \in U$, *then* f *is a constant.*

Proof. Without loss of generality, we can multiply a complex number of modulus one to make $f(z_0) = |f(z_0)|$. Let $S = \{z \in U \mid f(z) = f(z_0)\}$, then $S \neq \emptyset$ since $z_0 \in S$. Since f is continuous on U, S is a closed set (why?).

Based on the above discussion, if $w \in S$, there is a small disk $D(w, r) \subset U$. The Cauchy integral formula shows that for any $0 < r' < r$,

$$\begin{aligned} f(w) &= \frac{1}{2\pi i} \int_{\partial D(w,r')} \frac{f(z)}{z - w} dz = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(w + r'e^{i\theta})}{r'e^{i\theta}} ir'e^{i\theta} d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(w + r'e^{i\theta}) d\theta, \end{aligned}$$

which implies that

$$|f(w)| = \left|\frac{1}{2\pi}\int_0^{2\pi} f(w + r'e^{i\theta})d\theta\right| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(w + r'e^{i\theta})|d\theta.$$

Recall that $f(w) \geq |f(z)|$ for all $z \in U$, the equality holds and

$$f(w) = |f(w)| = f(w + r'e^{i\theta}), \quad \forall (r', \theta) \in (0, r) \times [0, 2\pi).$$

Therefore, S is an open set since a small open disk $D(w, r) \subset S$. Because S is a nonempty, open, and closed subset of a simply connected set U, we must have $S = U$. □

The maximum principle only uses the mean-value property. It means harmonic function also has the maximum modulus principle (consider the real/imaginary part of f). A very powerful corollary of this principle is the **Schwartz Lemma**.

Theorem 0.3.21 (Schwartz Lemma). *If f is holomorphic that maps the unit disk $D(0,1)$ to itself and $f(0) = 0$, then*

$$|f(z)| \leq |z|, \quad |f'(0)| \leq 1.$$

The equalities hold when $f(z) = e^{i\theta}z$ for some $\theta \in \mathbb{R}$.

Proof idea. The idea is to consider the holomorphic function $G(z) = f(z)/z$, since $f(0) = 0$, we can assign $G(0) = f'(0)$. By **maximum modulus principle**, on $D(0, 1 - \varepsilon)$,

$$|G(z)| \leq \frac{\max_{|z|=1-\varepsilon} |f(z)|}{1 - \varepsilon} < \frac{1}{1 - \varepsilon}.$$

Note that it is **strictly less than**, since f cannot achieve its maximum inside the unit disk. Let $\varepsilon \to 0$, we get $|G(z)| \leq 1$ for $z \in D(0, 1)$, that is, $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$. The equality holds when $G(z)$ is a constant of modulus one. □

0.4 Laurent Series

The major difference between real series and complex series is the **Laurent series**. This section mainly follows Weierstrass' theory.

0.4.1 Weierstrass Theories

The Weierstrass Theorem is a profound result in complex analysis.

Theorem 0.4.1 (Weierstrass Theorem). *Let $\{f_n(z)\}$ be a sequence of holomorphic functions in an open connected set $U \subset \mathbb{C}$. If the series $\sum_{n=1}^{\infty} f_n(z)$ converges uniformly to $f(z)$ on every compact subset of U, then f is holomorphic on U. Moreover, for all $k \in \mathbb{N}$,*

$$\sum_{n=1}^{\infty} f_n^{(k)} \to f^{(k)}$$

uniformly on every compact subset of U.

Proof Sketch. $\sum_{n=1}^{\infty} f_n$ uniformly converges to f, then on any simple closed curve $\gamma \subset U$,

$$\left|\int_\gamma f(z)dz\right| = \left|\int_\gamma \left(\sum_{n=1}^{N} f_n(z) - f(z)\right) dz\right| \leq \int_\gamma \left|\sum_{n=1}^{N} f_n(z) - f(z)\right| dz \xrightarrow{N\to\infty} 0.$$

Therefore,

$$\int_\gamma f(z)dz = 0.$$

By **Morera Theorem** (Corollary 0.3.9), f is holomorphic on U. The uniform convergence of derivatives comes from the **Taylor series** (Theorem 0.3.7),

$$\begin{aligned}\left|\sum_{n=1}^{N} f_n^{(k)}(\zeta) - f^{(k)}(\zeta)\right| &= \left|\frac{k!}{2\pi i}\int_{\partial D(\zeta,r)} \frac{\sum_{n=1}^{N} f_n(z) - f(z)}{(z-\zeta)^{k+1}} dz\right| \\ &\leq k! \sup_{z\in\partial D(\zeta,r)} \frac{\left|\sum_{n=1}^{N} f_n(z) - f(z)\right|}{r^k} \xrightarrow{N\to\infty} 0.\end{aligned}$$

□

The Laurent series is an extension of the Taylor series with *poles*. Let $z_0 \in \mathbb{C}$ and $c_n \in \mathbb{C}, n \in \mathbb{Z}$, a series of the form

$$\sum_{n=-\infty}^{\infty} c_n(z-z_0)^n = \underbrace{\sum_{n=0}^{\infty} c_n(z-z_0)^n}_{\text{holomorphic part}} + \underbrace{\sum_{n=1}^{\infty} c_{-n}(z-z_0)^{-n}}_{\text{principal part}}$$

is called *Laurent series* at z_0. The series distinguishes itself from the Taylor series in the second part (called *principal part*) by including negative exponents. If both parts converge at a point z, then we say the Laurent series is convergent at z.

If the first part converges in $|z - z_0| < R$, it will converge absolutely (Theorem 0.3.7) and uniformly on every compact subset. Therefore, the first part is holomorphic (Theorem 0.4.1).

For the second part, let $\zeta = 1/(z-z_0)$, it becomes

$$\sum_{n=1}^{\infty} c_{-n}(z-z_0)^{-n} = \sum_{n=1}^{\infty} c_{-n}\zeta^n.$$

If the radius of convergence of the series is $\lambda > 0$, then it converges absolutely in $|\zeta| < \lambda$ and uniformly on every compact subset, making it holomorphic in the disk as well. Thus, the Laurent series is holomorphic on $\lambda^{-1} < |z-z_0| < R$ once $R > \lambda^{-1}$. This region is called the *annulus of convergence*. If $\lambda^{-1} > R$, the Laurent series *diverges* everywhere. If $\lambda^{-1} = R$, the series diverges everywhere except on $|z-z_0| = R$. However, it does ***not*** mean the series converges everywhere on the circle.

Example 0.4.2. *The series*

$$\sum_{n=-\infty, n\neq 0}^{\infty} \frac{z^n}{|n|}$$

converges everywhere on $|z| = 1$ *except at* $z = 1$ *(prove it with the* ***Dirichlet Test****).*

An analogue of the Taylor series argument exists for the Laurent series, by considering the two parts separately. We omit the proof here.

Theorem 0.4.3. *If f is holomorphic on the annulus $r < |z-z_0| < R$, then*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z-z_0)^n$$

where

$$c_n = \frac{1}{2\pi i}\int_{|z-z_0|=\rho} \frac{f(z)}{(z-z_0)^{n+1}}dz, \quad r < \rho < R.$$

Moreover, it also implies the Laurent series of f is unique.

0.4.2 Isolated Singularity

If a function f is holomorphic on $D(z_0, R) - \{z_0\}$, that is, *a punctured disk*, the point z_0 is called **isolated singularity**. If z_0 is an isolated singularity, then the Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z-z_0)^n$$

is valid on $0 < |z - z_0| < R$ and the coefficients

$$c_n = \frac{1}{2\pi i} \int_{D(z_0,\rho)} \frac{f(z)}{(z - z_0)^{n+1}} dz.$$

Regarding the behavior of $\lim_{z\to z_0} f(z)$, there are only two cases.

1. $\lim_{z\to z_0} f(z)$ exists and is finite. Then consider the function $F(z) = (z - z_0)^2 f(z)$, we find that $F(z)$ is holomorphic on $D(z_0, R)$. Because of the finiteness of f around z_0,

$$\lim_{z\to z_0} \frac{F(z) - F(z_0)}{z - z_0} = \lim_{z\to z_0} \frac{F(z)}{z - z_0} = \lim_{z\to z_0} (z - z_0) f(z) = 0.$$

   Thus F is holomorphic, and the Taylor series is

$$F(z) = 0 + 0 \cdot (z - z_0) + a_2(z - z_0)^2 + a_3(z - z_0)^2 + \cdots$$

   Therefore, $\frac{F(z)}{(z-z_0)^2} = a_2 + a_3(z - z_0) + \cdots$ is the continuation of f to $D(z_0, R)$, which does not contain the principal part. In this case, the singularity is called a *removable singularity*.

2. If $\lim_{z\to z_0} f(z) = \infty$ (the limit exists), then consider $F(z) = 1/f(z)$ which is holomorphic in $0 < |z-z_0| < R$, since $\lim_{z\to z_0} F(z) = 0$, it is a removable singularity for F, thus $F(z) = (z - z_0)^m G(z)$ that $G(z_0) \neq 0$. Locally, the Taylor series of $\frac{1}{G}$ is

$$\frac{1}{G(z)} = b_0 + b_1(z - z_0) + \cdots, \quad b_0 \neq 0.$$

   Therefore,

$$\begin{aligned} f(z) &= \frac{1}{(z - z_0)^m G(z)} \\ &= \frac{b_0}{(z - z_0)^m} + \frac{b_1}{(z - z_0)^{m-1}} + \cdots + b_m + b_{m+1}(z - z_0) + \cdots \end{aligned}$$

   It shows that the Laurent series's principal part can only have a finite number of nonzero terms. The singularity z_0 is also called a *pole* of order m. If $m = 1$, it is called a *simple pole*.

It remains another situation that the limit does not exist, which corresponds to there being infinitely many nonzero terms in the principal part; this case is called *essential singularity*. The behavior of f around an essential singularity is more complicated.

Example 0.4.4. *$f(z) = e^{1/z}$ has an essential singularity at $z = 0$. The limit does not exist at $z = 0$ since $\lim_{z\to 0^+} f(z) = \infty$ and $\lim_{z\to 0^-} f(z) = 0$.*

0.4.3 Residue Theorem

Let f be a holomorphic function on $D(z_0, r) - \{z_0\}$ that z_0 is an **isolated singularity** of f. The residue of f at z_0 is defined by

$$\text{Res}(f, z_0) = \frac{1}{2\pi i} \int_{|z-z_0|=\rho} f(z)dz.$$

where $0 < \rho < r$. If f has its Laurent series as $\sum_{n=-\infty}^{\infty} c_n(z - z_0)^n$, then the residue is c_{-1}.

Theorem 0.4.5 (Residue). *Suppose that f is holomorphic on an open connected set $U - \{z_1, \cdots, z_n\}$ and is continuous on $\overline{U} - \{z_1, \cdots, z_n\}$. Then*

$$\int_{\partial U} f(z)dz = 2\pi i \sum_{k=1}^{n} \text{Res}(f, z_k).$$

The proof is simple by the Cauchy Integral Theorem 0.3.3, so omitted here. This theorem is helpful when dealing with definite integrals for which an antiderivative is not available. The core technique is calculating the definite integral along a suitable choice of contour paths.

Example 0.4.6. *We evaluate the following integral*

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x^2}.$$

Consider the function $f(z) = \frac{1}{1+z^2}$. This function is holomorphic except for simple poles at $\pm i$. We choose the contour shown in Figure 4. Then

$$\int_{\Gamma} f(z) = 2\pi i \,\text{Res}(f, i) = 2\pi i \cdot \frac{1}{2i} = \pi.$$

It remains to verify that the integral on the arc is bounded by $\frac{\pi R}{R^2-1} \to 0$ as $R \to \infty$ (prove it). The same contour can be used to calculate

$$\int_{-\infty}^{\infty} \frac{dx}{P(x)} = 2\pi i \sum_{k=1}^{n} \text{Res}(P, z_k),$$

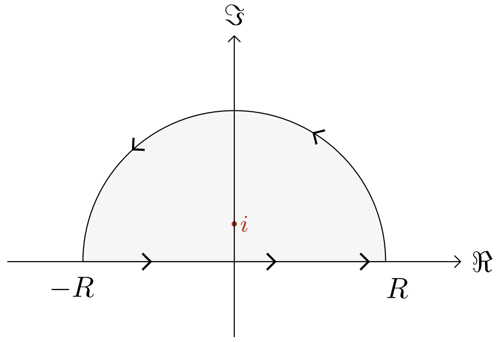

Figure 4: Contour on half circle.

where the polynomial P is strictly positive and $\deg(P) \geq 2$*,* $\{z_k\}_{k=1}^n$ *is the set of roots in the upper half plane. For instance, if* $P(x) = (1+x^2)^{n+1}$*,* $z = i$ *is a root of multiplicity* $n+1$*, then the residue is*

$$\frac{1}{n!}\frac{d^n}{dz^n}\left(\frac{(z-i)^{n+1}}{(1+x^2)^{n+1}}\right)\Bigg|_{z=i} = \frac{1}{n!}\frac{d^n}{dz^n}\frac{1}{(z+i)^{n+1}}\Bigg|_{z=i} = -i\frac{2n!}{n!n!}\frac{1}{2^{2n+1}}.$$

Then the integral is $\frac{\pi}{2^{2n}}\frac{2n!}{n!n!}$

Example 0.4.7. *Another example is the Dirichlet integral*

$$\int_{-\infty}^{\infty}\frac{\sin x}{x}$$

Choosing $f(z) = \frac{e^{iz}}{z}$ *(why not* $\frac{e^{-iz}}{z}$*?), then the contour is almost the same as the previous one (see Figure 5) with a small contour around* $z = 0$*, since it is a pole of order* 1*.*

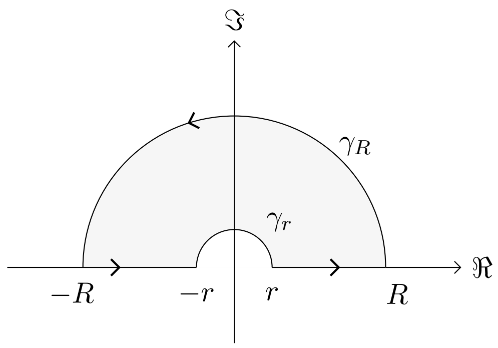

Figure 5: Contour on half circle.

Then

$$\left(\int_{-R}^{-r}+\int_{r}^{R}+\int_{\gamma_R}+\int_{\gamma_r}\right)\frac{e^{iz}}{z}dz=0.$$

Estimating the interals on γ_R should be careful since $\int_{\gamma_R}\frac{|dz|}{|z|}$ is not converging to zero as $R\to\infty$.

$$\begin{aligned}\int_{\gamma_R}\frac{e^{iz}}{z}dz&=\int_0^\pi\frac{e^{iR(\cos\theta+i\sin\theta)}}{Re^{i\theta}}iRe^{i\theta}d\theta\\&=i\int_0^\pi e^{iR\cos\theta-R\sin\theta}d\theta.\end{aligned}$$

The magnitude is bounded by (since $\frac{2}{\pi}\theta\geq\sin\theta\geq\theta$ on $(0,\frac{\pi}{2})$)

$$\int_0^\pi e^{-R\sin\theta}d\theta=2\int_0^{\pi/2}e^{-R\sin\theta}d\theta\leq2\int_0^{\pi/2}e^{-R\frac{2}{\pi}\theta}d\theta=\frac{\pi}{2R}(1-e^{-R})\xrightarrow{R\to\infty}0.$$

The integral on γ_r is (clockwise)

$$\int_{\gamma_r}\frac{e^{iz}}{z}dz=i\int_\pi^0 e^{ir\cos\theta-r\sin\theta}d\theta\xrightarrow{r\to0}=-i\pi.$$

Thus, by taking the imaginary part,

$$\int_{-\infty}^{\infty}\frac{\sin x}{x}dx=\pi.$$

Example 0.4.8. *Evaluate*

$$\int_0^\infty\sin(x^n)dx.$$

If we take the upper half circle as the contour, with $f(z)=e^{iz^n}$, the integral on the arc will be difficult to bound because $\Im(z^n)$ is not always positive $n\geq2$. Therefore, we need to restrict to a sector $\{0\leq\arg z\leq\frac{\pi}{2n}\}$ (see Figure 6).

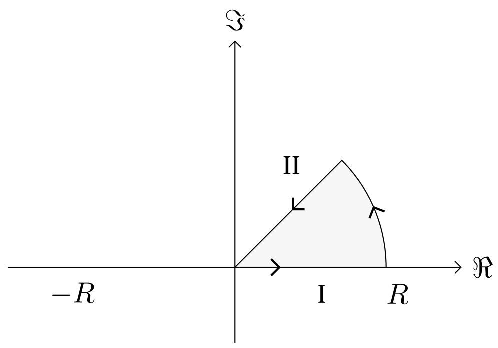

Figure 6: Contour path around a sector.

Then the integral becomes

$$\left(\int_I + \int_{II}\right) f(z)dz = 0.$$

That is,

$$\int_0^\infty e^{ix^n}dx = \int_0^\infty e^{ix^n e^{i\pi/2}} e^{i\pi/2n}dx$$
$$= e^{i\pi/2n}\int_0^\infty e^{-x^n}dx = e^{i\pi/2n}\Gamma(\frac{n+1}{n}).$$

The integral

$$\int_0^\infty e^{-x^n}dx = \int_0^\infty \exp(-u)\frac{1}{n}u^{1-\frac{1}{n}}du = \frac{1}{n}\Gamma(\frac{1}{n}) = \Gamma(1+\frac{1}{n}).$$

Therefore, the final result is $\Gamma(\frac{n+1}{n})\sin(\frac{\pi}{2n})$*. It also shows that the integral decays like* $\frac{\pi}{2n}$ *as* n *approaches infinity.*

Remark 0.4.9. *The idea of the above example is the same as the* ***steepest descent*** *idea for asymptotic approximation. For instance, if we need to estimate a contour integral for large* $\lambda > 0$*:*

$$\int_\gamma f(z)e^{\lambda g(z)}dz$$

The key idea is to search for another contour path such that $|e^{g(z)}| = e^{\Re g(z)}$ *increases as rapidly as possible, allowing the so-called Laplace method to be adopted. Let* $g(z) = u(z) + iv(z)$*, the Cauchy-Riemann equation shows that*

$$u_x v_x + u_y v_y = \nabla u \cdot \nabla v = 0.$$

It means the level sets of u *and* v *are orthogonal except for critical points. Having* $v \equiv c$ *means* u *is following a steepest descent path. For more details, see Chapter 8.*

Example 0.4.10 (Davis (1959))**.** *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a* 2π*-periodic analytic function that can be extended to the strip* $|\Im z| < b$ *such that* $|f| \le M$ *in the strip. Then, we can estimate the quadrature error of the trapezoid rule*

$$E_N = \frac{2\pi}{N}\sum_{j=1}^{N} f(2\pi j/N) - \int_0^{2\pi} f(x)dx.$$

Consider $F(z) = \cot(\frac{N}{2}z)f(z)$*, with a contour integral in the following Figure 7.*

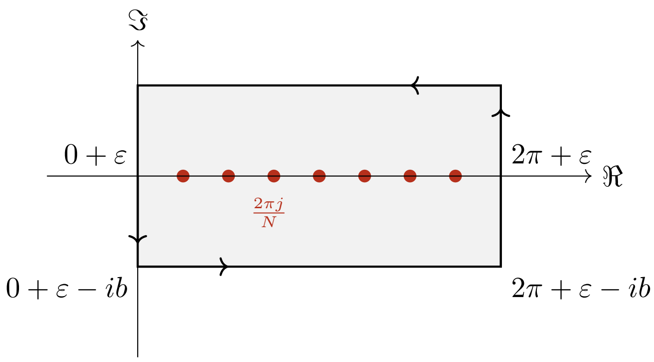

Figure 7: Contour for trapezoid rule.

Using the Residue Theorem (set $\varepsilon \to 0^+$)

$$\int_\gamma F(z)dz = 2\pi i \sum_{j=1}^{N} \text{Res}(F, \frac{2\pi j}{N}) = \frac{4\pi i}{N} \sum_{j=1}^{N} f(\frac{2\pi j}{N}).$$

The integrals on the two vertical sides cancel each other. Therefore, we obtain

$$\int_{0+\varepsilon-ib}^{2\pi+\varepsilon-ib} \cot(\frac{N}{2}z) f(z)dz - \int_{0+\varepsilon+ib}^{2\pi+\varepsilon+ib} \cot(\frac{N}{2}z) f(z)dz = \frac{4\pi i}{N} \sum_{j=1}^{N} f(\frac{2\pi j}{N}).$$

The two integrals are conjugate. We can write the left-hand side as

$$\Im \int_{0+\varepsilon-ib}^{2\pi+\varepsilon-ib} \cot(\frac{N}{2}z) f(z)dz = \frac{2\pi}{N} \sum_{j=1}^{N} f(\frac{2\pi j}{N}).$$

The Cauchy Integral Formula implies (why?)

$$\int_{0+\varepsilon-ib}^{2\pi+\varepsilon-ib} f(z)dz = \int_{0+\varepsilon}^{2\pi+\varepsilon} f(z)dz.$$

Therefore, we find

$$E_N = \Re \int_{0+\varepsilon-ib}^{2\pi+\varepsilon-ib} (-i \cot(\frac{N}{2}z) - 1) f(z)dz$$

Because $\cot\frac{N}{2}z = \frac{\sin(N\Re z) - i\sinh(N\Im z)}{\cosh(N\Im z)-\cos(N\Re z)}$, *see (DLMF, 2025, Equation 4.21.40), we obtain*

$$\begin{aligned}\left|-i\cot(\frac{N}{2}z)-1\right| &= \left|\frac{-i\sin(N\Re z)-\sinh(N\Im z)-\cosh(N\Im z)+\cos(N\Re z)}{\cosh(N\Im z)-\cos(N\Re z)}\right| \\ &\le \frac{1+\exp(N\Im z)}{\cosh(N\Im z)-1} = \frac{2+2e^{-Nb}}{e^{Nb}+e^{-Nb}-2} \le \frac{2}{e^{Nb}-1}.\end{aligned}$$

Thus, $|E_N| \le \frac{4\pi M}{e^{Nb}-1}$.

Remark 0.4.11. *This means doubling the number of quadrature nodes will roughly double the number of accurate digits.*

0.5 Exercises

Problem 0.5.1. *Show that* z_1, z_2, z_3 *are the vertices of an equilateral triangle on the complex plane* ***if and only if***

$$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1.$$

Problem 0.5.2. *Let* f *be a holomorphic function in an open connected set* U *containing the closure of a disk* $\overline{D(z_0, R)}$. *Show that*

$$|f^{(n)}(z_0)| \le \frac{n!}{R^n}\sup_{|\zeta|=1}|f(z_0+R\zeta)|.$$

Problem 0.5.3. *Suppose* $f \in C(U)$ *and the path integral of* f *along any* ***equilateral*** *triangle in* U *is zero. Prove* f *is holomorphic on* U.

Hint: If $f \in C^1(U)$, *prove it by Green's theorem. Then consider mollified* $f_\varepsilon = f * \psi_\varepsilon \in C^1(U)$ *and the sequence converges to* f.

Problem 0.5.4. *Prove Theorem 0.3.12.*

Problem 0.5.5. *Let* f *be a holomorphic function that maps the unit disk* $D(0,1)$ *to itself and* $f(0) = 0$. *Show that*

$$|f(z) + f(-z)| \le 2|z|^2.$$

Problem 0.5.6. *Let* f *be holomorphic on the unit disk* $D(0,1)$ *that* $f(0) = 0$ *and* $|\Re f(z)| < 1$ *for all* $z \in D(0,1)$. *Show that*

$$|\Re f(z)| \le \frac{4}{\pi}\arctan|z|, \quad |\Im f(z)| \le \frac{2}{\pi}\log\left(\frac{1+|z|}{1-|z|}\right) \quad \forall z \in D(0,1).$$

Hint: Consider the bijective map $\psi(z) = \frac{2}{\pi i}\log\frac{1+z}{1-z}$ *and apply Schwartz Lemma to* $\psi^{-1} \circ$

$f(z)$.

Problem 0.5.7. *Let $p_n(z) = a_0 + a_1 z + \cdots + a_n z^n$ be a polynomial of degree n. The coefficients $\{a_i\}_{i=0}^n$ are independently chosen from $[-1, 1]$ uniformly. Prove the probability that the smallest root of $p_n(z)$ is real is at least 2% for any $n \geq 1$.*

Problem 0.5.8. *Let $p_n(z) = (z - a_1) \cdots (z - a_n)$ such that $|a_i| = 1$, prove there exists a point $|\zeta| = 1$ that $|p_n(\zeta)| = 1$.*

Problem 0.5.9. *Evaluate the integrals*

- $\displaystyle\int_0^\infty \frac{1}{1 + x^n} dx$.
- $\displaystyle\int_0^\infty \frac{x^\alpha}{x^2 - 2x \cos\theta + 1} dx$
- $\displaystyle\int_0^\infty \frac{\ln^2 x}{x^2 + 1} dx$
- $\displaystyle\int_{-\infty}^\infty \frac{\cos(ax)}{1 + x^2} dx$

where $a > 0$ is a real number, $\alpha \in (0, 1)$, $\theta \in (0, \pi)$, $n \geq 2$ is an integer.

Extended Reading

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