Math 7000/7010 **Spring 2026**

Homework 5

Tags: Distribution & Green's Fuction *Due Date: 02/15/2026 11:59 CST*

1 Distribution

Problem 1.1. *Let* $u_\sigma(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{x^2}{2\sigma}}$. *Show that*

$$u_\sigma \xrightarrow{w} \delta(x), \quad \sigma \to 0^+.$$

Problem 1.2. *Prove the following statements.*

1. *Show that*

$$\frac{1}{2\pi i}\left(\frac{1}{x - i\varepsilon} - \frac{1}{x + i\varepsilon}\right) \xrightarrow{w} \delta(x), \quad \mathbb{R} \ni \varepsilon \to 0^+.$$

.

2. *Then use the above formula to evaluate*

$$\int_0^\infty \cos(xt) dt.$$

You may use

$$\mathrm{Re} \int_0^\infty e^{-ixt} dt = \mathrm{Re} \lim_{\varepsilon \to 0^+} \int_0^\infty e^{-ixt - \varepsilon t} dt$$

3. *Find a way (maybe not unique) to define* $D_\alpha(e^{-ixt})$ *as the* α*-th derivative of* e^{-ixt} *in the* x *variable, such that*

$$D_\alpha(D_\beta(e^{-ixt})) = D_{\alpha+\beta}(e^{-ixt}),$$

and $D_k(e^{-ixt}) = (-it)^k e^{-ixt}$ *for all* $k \in \mathbb{N}$.

4. *Find a way to represent the "$\frac{1}{2}$-derivative" of $\delta(x)$.*

2 Green's Function

Problem 2.1. *Let us solve the following equation*

$$\partial_x^{1/2} u = f(x)$$

where $\partial_x^{1/2}$ is a fractional derivative which is consistent with your previous definition.

1. *Show that $\mathcal{G}(x) = D_{-\frac{1}{2}}\delta(x)$ is a fundamental solution.*
2. *Evaluate $\mathcal{G}(x)$ (may use some complex integral results from last semester).*
3. *Use your $\mathcal{G}(x)$ to represent the solution u.*

Problem 2.2. *Construct Green's functions and use them to represent the solutions for the following boundary value problems.*

1. $u''(x) + u(x) = f(x)$, $x \in (0, 1)$, $u'(0) = u(0)$, $u'(1) = 3u(1)$.
2. $x^2 u''(x) + 2xu'(x) = f(x)$, $x \in (1, 2)$, $u'(1) = 0$, $u(2) + u'(2) = 0$.