

Math 7000/7010

Fall 2025

Homework 4

Tags: *Basic Functional Analysis*

Due Date: 11/30/2025 11:59 CST

1 General Vector Space

Recall the definition of a norm $\|\cdot\|: V \times \mathbb{R}_+ \cup \{0\}$ is a function satisfying

1. $\|\lambda x\| = |\lambda| \|x\|$
2. $\|x + y\| \leq \|x\| + \|y\|$
3. $\|x\| \geq 0$, the equal sign holds iff $x = \mathbf{0}$.

You can also see the difference from the definition of “metric”. The first scaling rule does not apply to metrics.

Inner product $\langle \cdot, \cdot \rangle$ is a function that maps $V \times V \rightarrow \mathbb{C}$ or $V \times V \rightarrow \mathbb{R}$ satisfying

1. $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$
2. $\langle x, y \rangle = \overline{\langle y, x \rangle}$
3. $\langle x, x \rangle \geq 0$, equal sign holds iff $x = \mathbf{0}$.

Problem 1.1. We know if V is a vector space with inner product $\langle \cdot, \cdot \rangle$, then

$$\|f\| := \sqrt{\langle f, f \rangle}$$

defines a norm.

Is the converse correct? If $\|\cdot\|$ is a norm on a vector space V , is it necessary to be defined by an inner product?

First, prove that if such an inner product exists that defines the norm, then

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2, \quad \forall x, y \in V. \quad (1.1)$$

Use this to tell (prove or construct counterexamples) whether there is an inner product that can represent each of the following norms.

1. Let $v \in \mathbb{R}^2$ and $\|v\|_1 = \sum_{i=1}^2 |v_i|$

2. Let $v \in \mathbb{R}^2$ and $\|v\|_\infty = \max_{1 \leq i \leq 2} |v_i|$.

The proof is easy; do the expansion.

Neither case can be defined by an inner product.

1. Let $x = (1, 0)$, $y = (0, 1)$, then

$$\|x + y\|_1^2 + \|x - y\|_1^2 = 8 \neq 4 = 2\|x\|_1^2 + 2\|y\|_1^2.$$

2. Let $x = (1, 0)$, $y = (0, 1)$, then

$$\|x + y\|_\infty^2 + \|x - y\|_\infty^2 = 2 \neq 4 = 2\|x\|_\infty^2 + 2\|y\|_\infty^2.$$

Problem 1.2. Let V be a vector space with a norm $\|\cdot\|$ that

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2, \quad \forall x, y \in V. \quad (1.2)$$

Define a function $\phi : V \times V \rightarrow \mathbb{R}$:

$$\phi(x, y) = \frac{1}{4} (\|x + y\|^2 - \|x - y\|^2).$$

1. Show that $\phi(x, x) = \|x\|^2$ and $\phi(x, y) = \phi(y, x)$.

2. Compute $\phi(x, y)$ in another way

$$\phi(x, y) = \frac{1}{4} (\|x + y\|^2 + \|z\|^2) - \frac{1}{4} (\|x - y\|^2 + \|z\|^2)$$

and use (1.2) from the right side to the left side to show

$$\phi(x, y) = \frac{1}{2} \phi(x - z, y) + \frac{1}{2} \phi(x + z, y), \quad \forall x, y, z \in V.$$

3. Use the above identity (by a change of variable) to show

$$\phi(x + y, z) = \phi(x, z) + \phi(y, z). \quad (1.3)$$

4. Show $\phi(-x, y) = -\phi(x, y)$ and $\phi(nx, y) = n\phi(x, y)$ for any $n \in \mathbb{N}$ (use (1.3) and induction).

5. Let $p, q \in \mathbb{N}$ and show

$$\phi\left(\frac{q}{p}x, y\right) = q\phi\left(\frac{x}{p}, y\right), \quad p\phi\left(\frac{x}{p}, y\right) = \phi(x, y).$$

Hence, $\phi\left(\frac{q}{p}x, y\right) = \frac{q}{p}\phi(x, y)$.

6. For any $r \in \mathbb{R}$, choose a sequence of rational numbers $\frac{q_n}{p_n} \rightarrow r$, show that

$$\lim_{n \rightarrow \infty} \phi\left(\frac{q_n}{p_n}x, y\right) = \phi(rx, y).$$

7. Prove ϕ defines an inner product.

1. Simple.

2. From right side to left side,

$$\frac{1}{4}(\|x + y\|^2 + \|z\|^2) = \frac{1}{8}(2\|x + y\|^2 + 2\|z\|^2) = \frac{1}{8}(\|x + y + z\|^2 + \|x + y - z\|^2)$$

and

$$\frac{1}{4}(\|x - y\|^2 + \|z\|^2) = \frac{1}{8}(2\|x - y\|^2 + 2\|z\|^2) = \frac{1}{8}(\|x - y + z\|^2 + \|x - y - z\|^2)$$

Therefore,

$$\phi(x, y) = \frac{1}{8}(\|x + y + z\|^2 - \|x - y + z\|^2) + \frac{1}{8}(\|x + y - z\|^2 - \|x - y - z\|^2) = \frac{1}{2}\phi(x + z, y) + \frac{1}{2}\phi(x - z, y).$$

3. Since $\phi(x, y) = -\phi(-x, y)$,

$$\phi(x, y) = \frac{1}{2}\phi(x + z, y) + \frac{1}{2}\phi(x - z, y).$$

Then

$$\phi(z, y) = \frac{1}{2}\phi(x + z, y) + \frac{1}{2}\phi(z - x, y) = \frac{1}{2}\phi(x + z, y) - \frac{1}{2}\phi(x - z, y).$$

Take the sum.

4. Easy.

5. Easy.

6. We need to show that (continuity) of ϕ ,

$$\lim_{x \rightarrow z} \phi(x, y) = \phi(z, y).$$

This is equivalent to (by its definition)

$$\lim_{x \rightarrow z} \|x + y\| - \|z + y\| = 0$$

By the triangle inequality of the norm (and linearity), we know this is true.

7. Now we can verify that ϕ satisfies the 1st rule of inner product. The other two rules are automatically true.

Notation: The vector space $C_c^\infty(\mathbb{R}^3)$ contains all *smooth* functions with *compact support*. *Smooth* means all derivatives exist everywhere, and *compact support* means that for each $f \in C_c^\infty(\mathbb{R}^3)$, there exists a radius $R > 0$, $f(x) = 0$ if $|x| > R$.

Problem 1.3. Define

$$V = \{(f_1, f_2, f_3) \mid f_i \in C_c^\infty(\mathbb{R}^3)\}.$$

1. Show V is a vector space.

2. Show the following way

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_{\mathbb{R}^3} \sum_{i=1}^3 f_i(\mathbf{x}) \overline{g_i(\mathbf{x})} d\mathbf{x}$$

defines an inner product on V , $\mathbf{f} = (f_1, f_2, f_3)$, and $\mathbf{g} = (g_1, g_2, g_3)$.

3. Let $u \in C_c^\infty(\mathbb{R}^3)$. Show that

$$\left\langle \frac{\mathbf{x}u(\mathbf{x})}{|\mathbf{x}|^2}, \nabla u(\mathbf{x}) \right\rangle = -\frac{1}{2} \left\langle \frac{\mathbf{x}u(\mathbf{x})}{|\mathbf{x}|^2}, \frac{\mathbf{x}u(\mathbf{x})}{|\mathbf{x}|^2} \right\rangle.$$

4. Use the same idea for proving the Cauchy-Schwartz inequality to prove Hardy's inequality

$$\left\langle \frac{\mathbf{x}u(\mathbf{x})}{|\mathbf{x}|^2}, \frac{\mathbf{x}u(\mathbf{x})}{|\mathbf{x}|^2} \right\rangle \leq 4 \left\langle \nabla u(\mathbf{x}), \nabla u(\mathbf{x}) \right\rangle$$

or equivalently,

$$\int_{\mathbb{R}^3} \frac{|u(\mathbf{x})|^2}{|\mathbf{x}|^2} d\mathbf{x} \leq 4 \int_{\mathbb{R}^3} |\nabla u(\mathbf{x})|^2 d\mathbf{x}$$

5. If the support of $u(x)$ is inside $\{|\mathbf{x}| < R\}$, prove the Poincaré inequality

$$\int_{\mathbb{R}^3} |u(\mathbf{x})|^2 d\mathbf{x} \leq 4R^2 \int_{\mathbb{R}^3} |\nabla u(\mathbf{x})|^2 d\mathbf{x}$$

1. By definition.

2. By definition.

3. Note u and its derivatives are supported in a ball B_R ,

$$\left\langle \frac{\mathbf{x}u(\mathbf{x})}{|\mathbf{x}|^2}, \nabla u(\mathbf{x}) \right\rangle = \int_{B_R} \frac{\mathbf{x} \cdot (u(\mathbf{x}) \nabla u(\mathbf{x}))}{|\mathbf{x}|^2} d\mathbf{x}$$

The dot \cdot is the usual inner product in \mathbb{R}^3 .

By Green's theorem, we write $u(\mathbf{x})\nabla u(\mathbf{x}) = \frac{1}{2}\nabla|u(\mathbf{x})|^2$, then

$$\begin{aligned} \int_{B_R} \frac{\mathbf{x} \cdot (u(\mathbf{x})\nabla u(\mathbf{x}))}{|\mathbf{x}|^2} d\mathbf{x} &= \frac{1}{2} \int_{B_R} \frac{\mathbf{x}}{|\mathbf{x}|^2} \cdot \nabla(|u(\mathbf{x})|^2) d\mathbf{x} \\ &= \frac{1}{2} \int_{B_R} \nabla \cdot \left[\frac{\mathbf{x}}{|\mathbf{x}|^2} \cdot (|u(\mathbf{x})|^2) \right] d\mathbf{x} - \frac{1}{2} \int_{B_R} \left(\nabla \cdot \frac{\mathbf{x}}{|\mathbf{x}|^2} \right) |u(\mathbf{x})|^2 d\mathbf{x} \\ &= \frac{1}{2} \int_{\partial B_R} \mathbf{n}(\mathbf{x}) \cdot \left[\frac{\mathbf{x}}{|\mathbf{x}|^2} \cdot (|u(\mathbf{x})|^2) \right] dS - \frac{1}{2} \int_{B_R} \frac{1}{|\mathbf{x}|^2} (|u(\mathbf{x})|^2) d\mathbf{x} \\ &= 0 - \frac{1}{2} \int_{B_R} \frac{1}{|\mathbf{x}|^2} (|u(\mathbf{x})|^2) d\mathbf{x}. \end{aligned}$$

4. Let $v(t) = \frac{\mathbf{x}u(\mathbf{x})}{|\mathbf{x}|^2} + t\nabla u(\mathbf{x})$. Then

$$\begin{aligned} 0 \leq |v(t)|^2 &= \left\langle \frac{\mathbf{x}u(\mathbf{x})}{|\mathbf{x}|^2}, \frac{\mathbf{x}u(\mathbf{x})}{|\mathbf{x}|^2} \right\rangle + t^2 \langle \nabla u(\mathbf{x}), \nabla u(\mathbf{x}) \rangle + 2t \left\langle \frac{\mathbf{x}u(\mathbf{x})}{|\mathbf{x}|^2}, \nabla u(\mathbf{x}) \right\rangle \\ &= (1-t) \left\langle \frac{\mathbf{x}u(\mathbf{x})}{|\mathbf{x}|^2}, \frac{\mathbf{x}u(\mathbf{x})}{|\mathbf{x}|^2} \right\rangle + t^2 \langle \nabla u(\mathbf{x}), \nabla u(\mathbf{x}) \rangle \end{aligned}$$

The determinant ≤ 0 , therefore

$$\left[\left\langle \frac{\mathbf{x}u(\mathbf{x})}{|\mathbf{x}|^2}, \frac{\mathbf{x}u(\mathbf{x})}{|\mathbf{x}|^2} \right\rangle \right]^2 - 4 \langle \nabla u(\mathbf{x}), \nabla u(\mathbf{x}) \rangle \left\langle \frac{\mathbf{x}u(\mathbf{x})}{|\mathbf{x}|^2}, \frac{\mathbf{x}u(\mathbf{x})}{|\mathbf{x}|^2} \right\rangle \leq 0.$$

Thus

$$\left\langle \frac{\mathbf{x}u(\mathbf{x})}{|\mathbf{x}|^2}, \frac{\mathbf{x}u(\mathbf{x})}{|\mathbf{x}|^2} \right\rangle \leq 4 \langle \nabla u(\mathbf{x}), \nabla u(\mathbf{x}) \rangle$$

5. Just notice

$$\left\langle \frac{\mathbf{x}u(\mathbf{x})}{|\mathbf{x}|^2}, \frac{\mathbf{x}u(\mathbf{x})}{|\mathbf{x}|^2} \right\rangle = \int_{B_R} \frac{|u(\mathbf{x})|^2}{|\mathbf{x}|^2} d\mathbf{x} \geq \frac{1}{R^2} \int_{B_R} |u(\mathbf{x})|^2 d\mathbf{x}$$

Problem 1.4. Let $A \in \mathbb{C}^{n \times n}$ and $\rho(A) < 1$. Show $\lim_{n \rightarrow \infty} A^n = \mathbf{0}$.

Prove it for a diagonal matrix. Then reduce any matrix to its Jordan form and use a diagonal perturbation to make it diagonalizable.

Note: Similarity transforms do not change the trace of a matrix, that is

$$\operatorname{tr}(PAP^{-1}) = \operatorname{tr}(A).$$

Problem 1.5. If A is a positive matrix. Show that

$$\rho(A) = \lim_{n \rightarrow \infty} (\operatorname{tr}(A^n))^{1/n}.$$

First, $\operatorname{tr}(A^n) = \sum_{k=1}^m \lambda_k^n$. Since A is a positive matrix, its spectral radius is a simple eigenvalue; therefore, all other eigenvalues are strictly smaller (in magnitude). Thus

$$\left(\sum_{k=1}^m \lambda_k^n\right)^{1/n} = \rho(A) \left(1 + \sum_{i=2}^m \frac{\lambda_i^n}{\rho(A)^n}\right)^{1/n} \rightarrow \rho(A).$$