Math 7000/7010 **Fall 2025**

Homework 3

Tags: Perturbation Methods *Due Date: 10/30/2025 11:59 CST*

1 Regular Perturbation

Problem 1.1. *Find the first three terms in the asymptotic solution ansatz* $y = y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \cdots$ *for*

$$y'' + \varepsilon y' + \varepsilon^2 y = 0, \quad y(0) = 0, \quad y(1) = 1.$$

At $\mathcal{O}(\varepsilon^0)$,

$$y_0'' = 0, \quad y_0(0) = 0, \quad y_0(1) = 1$$

Thus, $y_0(x) = x$.

At $\mathcal{O}(\varepsilon^1)$,

$$y_1'' + y_0' = 0, \quad y_1(0) = 0, \quad y_1(1) = 0$$

Thus, $y_1(x) = -\frac{1}{2}x(x-1)$.

At $\mathcal{O}(\varepsilon^2)$,

$$y_2'' + y_1' + y_0 = 0, \quad y_2(0) = 0, \quad y_2(1) = 0$$

Thus, $y_2(x) = -\frac{1}{4}x^2 + \frac{1}{4}$.

2 Singular Perturbation

Problem 2.1. *Let* $0 < \varepsilon \ll 1$. *Find the asymptotic solution for*

$$\varepsilon y'' + p(t)y' + q(t)y = 0, \quad y(0) = a, \quad y(1) = b.$$

where $p(t) < 0$ *on* $[0, 1]$.

The boundary layer is at $t = 1$, if there is any.

Outer: $p(t)y' + q(t)y = 0$ with $y(0) = a$. The solution $y_{out}(t) = ae^{-\int_0^t \frac{q(s)}{p(s)}ds}$.

Inner: Introduce $s = \frac{t-1}{\delta}$ (note $s < 0$), and let $Y(s) = y_{inn}(t)$, then

$$\frac{\varepsilon}{\delta^2}Y''(s) + p(t)\frac{Y'(s)}{\delta} + q(t)Y(s) = 0.$$

The leading order must be $\delta = \varepsilon$, then $Y(s) = Y_0(s) + \varepsilon Y_1(s) + \cdots$ leads to

$$Y_0''(s) + p(1 + s\varepsilon)Y_0'(s) = 0.$$

Then

$$Y_0(s) = C_0 + C_1 e^{-\frac{1}{\varepsilon}\int_0^s p(1+\varepsilon u)du}.$$

and $Y(0) = b$, thus $Y_0(s) = C_0 + (b - C_0)e^{-\frac{1}{\varepsilon}\int_0^s p(1+u\varepsilon)du}$.

Match: let $t = 1 - A\varepsilon^\beta$, $0 < \beta < 1$, then

$$y_{out}(t) \xrightarrow[\varepsilon\to 0]{} ae^{-\int_0^1 \frac{q(s)}{p(s)}ds}$$

and

$$y_{inn}(t) \to Y_0(-\infty) = C_0 = ae^{-\int_0^1 \frac{q(s)}{p(s)}ds}$$

Thus,

$$y(t) \sim ae^{-\int_0^t \frac{q(s)}{p(s)}ds} + (b - ae^{-\int_0^1 \frac{q(s)}{p(s)}ds})e^{-\frac{1}{\varepsilon}\int_0^{(t-1)/\varepsilon} p(1+\varepsilon u)du}$$

Problem 2.2. *Let* $0 < \varepsilon \ll 1$. *Find the asymptotic solution for*

$$\varepsilon^2 y'' + 2\varepsilon p(x)y' - q(x)y = f(x), \quad y(0) = a, \quad y(1) = b.$$

The functions p, q, f *are continuous, and* q *is positive.*

The outer solution $-q(x)y_{out}(x) = f(x)$, $y_{out}(x) = -\frac{f(x)}{q(x)}$. The outer solution does NOT need any boundary condition.

Thus, this solution may not agree with the boundary values. We may assume two boundary layers, then. The layer behaviors depend on how the scaling looks within the layers.

Are there any interior layers? We need to know which layers should be connected to the outer solution(s). If there is an interior layer, the outer solutions on both sides must differ.

Inner (left): $Y(s) = y(x)$, with $s = \frac{x}{\delta}$, then

$$\varepsilon^2\frac{Y''(s)}{\delta^2} + 2\varepsilon p(\delta s)\frac{Y'}{\delta} - q(s\delta)Y = f(s\delta)$$

There are three kinds of terms; the only possible choice is:

$$\frac{\varepsilon^2}{\delta^2} \sim \frac{\varepsilon}{\delta} \Rightarrow \varepsilon = \delta$$

otherwise $\frac{\varepsilon}{\delta} = o(1)$, leading to outer solution. Then

$$Y''(s) + 2p(s\varepsilon)Y' - q(s\varepsilon)Y = f(s\varepsilon).$$

and $Y(s) = Y_0(s) + \varepsilon Y_1(s) + \cdots$, the $\mathcal{O}(1)$ term (after Taylor expansion),

$$Y_0''(s) + 2p(0)Y_0' - q(0)Y_0 = f(0)$$

Since $q(0) > 0$, the equation $\lambda^2 + 2p(0)\lambda - q(0) = 0$ has two roots $\lambda_1 > 0$ and $\lambda_2 < 0$.

For the layer solution Y_0, it has two boundary conditions: one from $Y_0(0) = a$ and $Y_0(\infty) = y_{out}(0)$. So uniquely determined.

Inner (right): Let $Z(s) = y(x)$, $s = \frac{x-1}{\delta}$, then $\delta = \varepsilon$ and

$$Z''(s) + 2p(1 + s\delta)Z' - q(1 + s\delta)Z = f(1 + s\delta).$$

The leading term

$$Z_0'' + 2p(1)Z_0' - q(1)Z_0 = f(1)$$

Two boundary conditions are $Z_0(0) = y(1) = b$ and $Z(-\infty) = y_{out}(1)$.

Problem 2.3. *Let* $0 < \varepsilon \ll 1$. *Find the asymptotic solution for*

$$\varepsilon y'' + \left(\frac{1}{2} - t\right) y' + y = 0, \quad y(0) = 0, \quad y(1) = 1.$$

The layer is at $t = 0$ and (or) $t = 1$.

We can modify the problem as

$$\varepsilon z'' + \left(\frac{1}{2} - t\right) z' + z = 0, \quad z(0) = \frac{1}{2}, \quad z(1) = \frac{1}{2}.$$

The solution will be $y(t) = z(t) + (t - \frac{1}{2})$. The reason we do this is that the flipped solution w

$$w(t) := z(1 - t)$$

satisfies

$$\varepsilon w''(t) + (t - \frac{1}{2})(-1)w'(t) + w(t) = 0, \quad w(0) = w(1) = \frac{1}{2}.$$

Therefore, $w(t) = z(t)$ since the equation is the same. That means z should be symmetric.

Outer solution: $(\frac{1}{2}-t)z'_{out}+z_{out}=0$, $z_{out}=C(t-\frac{1}{2})$. This should be symmetric; the only choice is that $C=0$.

Inner (left): $Z(s)=z(t)$, $s=\frac{t}{\delta}$,

$$\varepsilon\frac{Z''}{\delta^2}+(\frac{1}{2}-s\delta)\frac{Z'}{\delta}+Z=0$$

Then $\varepsilon=\delta$. The first term: $Z_0''(s)+\frac{1}{2}Z_0'=0$, $Z_0(s)=C_0+C_1e^{-\frac{1}{2}s}$. $Z_0(0)=\frac{1}{2}$ and $Z_0(\infty)=0$. This means

$$C_0+C_1=\frac{1}{2},\quad 0=C_0\Rightarrow C_1=\frac{1}{2}.$$

Inner (right): Flip the solution.

Therefore, $y(t)=z(t)+(t-\frac{1}{2})\sim\frac{1}{2}e^{-x/(2\varepsilon)}+(t-\frac{1}{2})+\frac{1}{2}e^{-(1-x)/(2\varepsilon)}$.

Problem 2.4. *Let* $0<\varepsilon\ll 1$. *Find the asymptotic solution for*

$$\varepsilon y''+\varepsilon(x+1)^2y'-y=x-1,\quad y(0)=0,\quad y(1)=-1.$$

Outer: $y_{out}(x)=1-x$.

Inner (left): $Y(s)=y(x)$

$$\varepsilon\frac{Y''}{\delta^2}+\varepsilon(\delta s+1)^2\frac{Y'}{\delta}-Y=\delta s-1.$$

There are 4 terms $\frac{\varepsilon}{\delta^2}$, $\frac{\varepsilon}{\delta}$, 1, δ.

There are 6 cases for the two leading terms.

1. $\varepsilon/\delta^2=\varepsilon/\delta$, outer solution.
2. $\varepsilon/\delta^2=1$, $\delta=\sqrt{\varepsilon}$. This is fine.
3. $\varepsilon/\delta^2=\delta$, $\delta=\varepsilon^{1/3}$, this is not as large as 1.
4. $\varepsilon/\delta=1$, $\varepsilon=\delta$, this is not as large as ε/δ^2.
5. $\varepsilon/\delta=\delta$, $\delta=\sqrt{\varepsilon}$. This is not as large as 1.
6. $1=\delta$, outer.

Then, $\delta=\sqrt{\varepsilon}$.

$$Y''(s)+\sqrt{\varepsilon}(1+\sqrt{\varepsilon}s)^2Y'-Y=\sqrt{\varepsilon}s-1$$

The solution $Y(s) \sim Y_0(s) + \varepsilon^{1/2} Y_{1/2}(s) + \cdots$, we need the next order to be $\varepsilon^{1/2}$ otherwise we cannot cancel. Then

$$Y_0''(s) - Y_0(s) = -1$$

$Y_0(0) = 0$, $Y_0(\infty) = 1$. Thus $Y_0(s) = 1 - \exp(-s)$.

The layer solution at $x = 1$ can be derived similarly.

Problem 2.5. *Let* $0 < \varepsilon \ll 1$. *Find the asymptotic solution for*

$$\varepsilon y'' - y(1+y)y' - 3y = 0, \quad y(0) = 2, \quad y(1) = 2.$$

If we believe the solution does not change sign, then $y > 0$, which means the layer is on the right boundary point.

Outer: $(1+y)y' = -3$, thus $y_{out}(t) = \sqrt{C - 6t} - 1$.

First, we need to see whether we can have an interior layer. If there is, then on the left side,

$$\sqrt{C_L} - 1 = 2 \Rightarrow C_L = 9$$

On the right side

$$\sqrt{C_R - 6} - 1 = 2 \Rightarrow C_R = 15$$

Both solutions are positive on $[0, 1]$, which does not allow an interior layer since there is no sign change.

If the layer is on $x = 1$, then the boundary condition $y_{out}(0) = 2$ is valid, $C = 9$.

Let $s = \frac{x-1}{\delta}$,

$$\varepsilon \frac{Y''}{\delta^2} - Y(s)(1 + Y(s)) \frac{Y'(s)}{\delta} - 3Y = 0$$

The only choice is $\delta = \varepsilon$, thus

$$Y_0'' - Y_0(1 + Y_0)Y_0' = 0$$

Let $r(Y_0) = Y_0'(s)$, then

$$r'r = Y_0(1 + Y_0)r \Rightarrow r'(Y_0) = Y_0(1 + Y_0)$$

Then $r(Y_0) = \frac{Y_0^2}{2} + \frac{Y_0^3}{3} + C_0$. And

$$Y_0'(s) = \frac{Y_0^2}{2} + \frac{Y_0^3}{3} + C \Rightarrow \int \frac{dY_0}{\frac{Y_0^2}{2} + \frac{Y_0^3}{3} + C_0} = s + C_1$$

And $Y_0(0) = 2$, $Y_0(-\infty) = \sqrt{3} - 1$.

We must have $\lambda_1 = \sqrt{3} - 1$ as a root of $\frac{1}{3}\lambda^3 + \frac{\lambda^2}{2} + C_0 = 0$,

$$\frac{1}{3}(\sqrt{3}-1)^3 + \frac{1}{2}(\sqrt{3}-1)^2 = -C_0 \Rightarrow C_0 = \frac{1}{3}(4 - 3\sqrt{3}).$$

Thus, let $A = \frac{1+2\sqrt{3}}{4}$, $B = \frac{18-5\sqrt{3}}{8}$.

$$\int \frac{dY_0}{\frac{Y_0^2}{2} + \frac{Y_0^3}{3} + C_0} = \frac{3}{\lambda_1^2 + B}\left(\int \frac{dY_0}{Y_0 - \lambda_1} + \int \frac{(-Y_0 - 2A - \lambda_1)dY_0}{((Y_0 + A)^2 + B)}\right)$$

Then use $Y_0(0) = 2$ to determine the constant C_1.