Math 7000/7010 **Fall 2025**

Homework 2

Tags: Asymptotic Approximation *Due Date: 10/02/2025 11:59 CST*

1 Asymptotic Expansion

Problem 1.1. *For what values of α (if exists), yield $f(x) = \mathcal{O}(x^{\alpha})$ as $x \to 0^+$.*

1. $f(x) = \sqrt{1+x^3}$
2. $f(x) = x\sin(x)$
3. $f(x) = \dfrac{1}{1-e^x}$
4. $f(x) = \sin(1/x)$

1. $\alpha \le 0$. $x \to 0$ means $f(x) \to 1$.
2. $\alpha \le 2$. Use Taylor expansion.
3. $\alpha \le -1$. Use Taylor expansion.
4. $\alpha \le 0$. Since $\sin(1/x)$ visits 1 arbitrarily many times as $x \to 0^+$.

Problem 1.2. *Let $\{1, x, x^2, \cdots\}$ be a sequence of gauge functions, $x \to 0^+$, find the asymptotic expansion of*

$$f(x) = \int_0^1 e^{xt^2}\,dt$$

using the sequence.

$e^{xt^2} = \sum_{k\ge 0} \frac{x^k t^{2k}}{k!}$, then the integral becomes

$$\sum_{k\ge\infty} \frac{1}{2k+1} x^k.$$

Problem 1.3. *Let* $\{x^{-1}, 1, x, x^2, \cdots\}$ *be a sequence of gauge functions,* $x \to 0^+$*. If we try to find the asymptotic expansion of*

$$f(x) = \int_0^1 \frac{1}{x^2+t^2} dt.$$

Using the Taylor expansion idea, we will get

$$f(x) \sim \int_0^1 \left(\frac{1}{t^2} - \frac{x^2}{t^4} + \frac{x^4}{t^6} - \cdots \right) dt$$

while the terms are not integrable on $[0, 1]$ *due to the singularity at* $t = 0$*. This is because the Taylor expansion is only valid for* $x < t$*. For* $t < x$*, the integral needs to be handled in other ways.*

1. *find the asymptotic expansion in the form* $\frac{A}{x} + B + Cx + Dx^2 + Ex^3$ *of*

   $$f(x) = \int_\delta^1 \frac{1}{x^2+t^2} dt,$$

   The cutoff point δ *is chosen much larger than* x*.*

2. *Use a change of variable* $z = \frac{\delta}{t}$*, rewrite the integral*

   $$\int_0^\delta \frac{1}{x^2+t^2} dt$$

   Find the asymptotic expansion of the form $\frac{a}{x} + b + cx + dx^2 + ex^3$*.*

3. *Combine the two asymptotic expansions.*

1.

$$\int_\delta^1 \frac{1}{x^2+t^2} dt = \int_\delta^1 \left(\frac{1}{t^2} - \frac{x^2}{t^4} + \frac{x^4}{t^6} - \cdots \right) dt = (\frac{1}{\delta} - 1) + x^2(\frac{1}{3} - \frac{1}{3\delta^3}) + x^4(\frac{1}{5} - \frac{1}{5\delta^5}) + \cdots$$

2.

$$\int_0^\delta \frac{1}{x^2+t^2} dt = \frac{1}{x} \int_0^{\delta/x} \frac{1}{1+u^2} du = \frac{1}{x} \arctan(\delta/x) = \frac{1}{x} \left(\frac{\pi}{2} - (x/\delta) + \frac{1}{3}(x/\delta)^3 - \frac{1}{5}(x/\delta)^5 + \cdots \right)$$

3. Combine them.

$$\frac{\pi}{2x} - 1 + \frac{x^2}{3} + \frac{x^4}{5} + \cdots$$

2 Laplace Method

Problem 2.1. *Use the Laplace method to find the asymptotic approximation of*

$$\int_0^\infty \frac{e^{\lambda(2t-t^2)}}{1+t^2}dt, \quad \lambda \to -\infty.$$

$\phi(t) = 2t - t^2$, $\phi'(t_0) = 0$ means $t_0 = 1$, $\phi''(t_0) = -2$. The asymptotic approximation is

$$\sqrt{\frac{2\pi}{|\lambda||\phi''(t_0)|}} e^{\lambda\phi(t_0)} \frac{1}{1+t_0^2} = \sqrt{\frac{\pi}{|\lambda|}} \frac{1}{2} e^{\lambda}$$

Here λ needs to be negative, otherwise the integral does not converge.

Problem 2.2. *Find the asymptotic approximation for the integral*

$$I(\lambda) = \int_0^1 t^{\lambda k}(1-t)^{\lambda m} dt$$

as $\lambda \to \infty$, k, m are positive real numbers.

Convert to

$$\int_0^1 e^{\lambda[k \ln t + m \ln(1-t)]} dt$$

$\phi(t) = k \ln t + m \ln(1-t)$, $\phi'(t_0) = \frac{k}{t_0} - \frac{m}{1-t_0} = 0$, solves $t_0 = \frac{k}{m+k}$, and

$$\phi''(t_0) = -\frac{k}{t_0^2} - \frac{m}{(1-t_0)^2} = -\frac{(m+k)^3}{mk}.$$

The integral's approximation is

$$\sqrt{\frac{2\pi mk}{\lambda(m+k)^3}} (\frac{k}{m+k})^{\lambda k} (\frac{m}{m+k})^{\lambda m}.$$

Problem 2.3. *Find the asymptotic approximation for*

$$I(\lambda) = \int_0^\infty e^{-\lambda t} e^{-1/t^2} dt$$

Let $t = \lambda^{-1/3}u$, then

$$\lambda^{-1/3}\int_0^\infty e^{-\lambda^{2/3}(u+u^{-2})}du$$

$\phi(u) = u + u^{-2}$ has stationary point at $u_0 = 2^{1/3}$ and $\phi''(u_0) = \frac{6}{u_0^4} = \frac{3}{2^{1/3}}$. The approximation is

$$\lambda^{-1/3}\sqrt{\frac{2\pi 2^{1/3}}{\lambda^{3/2}3}}e^{-3\lambda^{2/3}/(2^{2/3})}$$

3 Watson's Lemma

Problem 3.1. *Find the first three terms of the asymptotic approximation for*

$$\int_0^1 e^{-\lambda \sin^4 t}\frac{t}{1+t^2}dt$$

as $\lambda \to \infty$.

You can start with a change of variable $u = \sin^4 t$*, and the Taylor series of* $\arcsin(x) = x + \frac{x^3}{6} + \frac{3x^5}{40} + \cdots$

After change of variable $u = \sin^4(t)$, $t = \arcsin(u^{1/4})$,

$$\int_0^1 e^{-\lambda u}\frac{\arcsin u^{1/4}}{1+\arcsin(u^{1/4})^2}\frac{1}{\sqrt{1-u^{1/2}}}\frac{1}{4}u^{-3/4}du \approx \int_0^1 e^{-\lambda u}\left(u^{1/4} - \frac{1}{2}u^{3/4} + \frac{13}{24}u^{5/4}\right)\frac{1}{4}u^{-3/4}du$$
$$\approx \frac{1}{4}\Gamma(\frac{1}{2})\lambda^{-1/2} - \frac{1}{8}\lambda^{-1} + \frac{13}{96}\Gamma(\frac{3}{2})\lambda^{-3/2}$$

4 Stationary Phase Theorem

Problem 4.1. *Suppose* $\phi \in C^\infty$ *does not have a stationary point on* (a, b)*. Let* $f(t)$ *be a smooth function on* $[a, b]$ *such that* $f^{(k)}(a) = f^{(k)}(b)$ *for all* $k \geq 0$*. Show that*

$$\int_a^b f(t)e^{i\lambda\phi(t)}dt = \mathcal{O}(\lambda^{-N}).$$

for any $N > 0$.

Use integration by parts,

$$\int_a^b f(t)e^{i\lambda\phi(t)}dt = \int_a^b f(t)\frac{\frac{d}{dt}e^{i\lambda\phi(t)}}{i\lambda\phi'(t)}dt = f(t)\frac{1}{i\lambda\phi'(t)}e^{i\lambda\phi(t)}|_a^b - \frac{1}{i\lambda}\int_a^b \frac{f(t)}{\phi'(t)}e^{i\lambda\phi(t)}dt$$

The first term is zero, then repeat integration by parts N times.

Problem 4.2. *The Fourier transform of a function f is defined by*

$$\widehat{f}(\zeta) := \int_{-\infty}^{\infty} f(x)e^{-ix\zeta}dx$$

Find the asymptotic approximation for the Fourier transform of the function $f(x) = e^{ix^k}$, $k \geq 2$ is an integer, as $\zeta \to \infty$.

Let $x = \zeta^{1/(k-1)}u$, $\lambda = \zeta^{k/(k-1)}$,

$$\int_{-\infty}^{\infty} e^{ix^k - ix\zeta}dx = \zeta^{1/(k-1)}\int_{-\infty}^{\infty} e^{i\lambda(u^k-u)}du$$

Stationary point $u_0 = k^{1/(k-1)}$, and Hessian is $k(k-1)k^{(k-2)/(k-1)} > 0$, the approximation is

$$\zeta^{1/(k-1)}\sqrt{\frac{2\pi}{\lambda k(k-1)k^{(k-2)/(k-1)}}}e^{i\lambda(k-1)k^{1/(k-1)}}e^{i\frac{\pi}{4}}$$