Math 7000/7010 **Fall 2025**

Homework 1

Tags: Complex Analysis *Due Date: 09/11/2025 11:59 CST*

1 Review: Complex Analysis

Problem 1.1. *Let $f(z)$ be a holomorphic function on an open connected set U containing the closure of a disk, namely $\overline{D(z_0, R)}$. Show that*

$$|f^{(n)}(z_0)| \leq \frac{n!}{R^n} \sup_{|z|=1} |f(z_0 + R\zeta)|.$$

The Cauchy integral formula

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\partial D} \frac{f(z)}{(z - z_0)^{n+1}} dz$$

Then

$$|f^{(n)}(z_0)| \leq \frac{1}{2\pi} \int_{\partial D} \frac{|f(z)|}{|z - z_0|^n} |dz| = n! \frac{2\pi R}{2\pi R^{n+1}} \sup_{|\zeta|=1} |f(z_0 + R\zeta)|.$$

Problem 1.2. *Suppose $f(z)$ be a holomorphic function in $0 < |z - z_0| < R$ and has Laurent series*

$$f(z) = \sum_{n=-m}^{\infty} c_n (z - z_0)^n,$$

where m is a finite number. Let k represent the number of roots of f inside $D(z_0, \rho)$, $\rho < R$. Compute

$$\frac{1}{2\pi i} \oint_{|z-z_0|=\rho} \frac{f'(z)}{f(z)} dz - k.$$

Let $f(z) = (z - z_0)^{-m} g(z)$, then

$$\frac{1}{2\pi i} \oint_{|z-z_0|=\rho} \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i} \oint_{|z-z_0|=\rho} \frac{(-m)}{z - z_0} dz + \frac{1}{2\pi i} \oint_{|z-z_0|=\rho} \frac{g'(z)}{g(z)} dz = k - m.$$

Problem 1.3. *Let* $f(z) = (z-a_1)\cdots(z-a_n)$*, such that* $|a_k| = 1$*,* $k = 1, 2, \cdots, n$*. Use the* ***maximum modulus theorem*** *to show* $\sup_{|z|=1}|f(z)| > 1$*.*

At $z = 0$, $|f(0)| = |\prod_{i=1}^{n}(-a_i)| = 1$, the maximum modulus theorem says $\max_{|z|=1}|f(z)| > |f(w)|$ for any $|w| < 1$, otherwise f is constant which is not possible.

Problem 1.4. *Let* $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$*, where* $a_0, \cdots a_n$ *are independently chosen from* $[-1, 1]$ *uniformly,* n *is a large degree. Source of the problem: math.stackexchange.com*

1. *If* $f(w) = 0$*, show that* $f(\overline{w}) = 0$*. In other words, complex roots come in pairs.*
2. *Let* $g(z) = a_1 z$*, show that*
$$|f(z) - g(z)| < |g(z)|$$
on the circle $|z| = \frac{1}{4}$ *when* $|a_1| > \frac{1}{2}$ *and* $|a_0| < \frac{1}{25}$*.*
3. *Show the root of the smallest modulus of* $f(z)$ *has at least 2% probability to be real.*

If a_k are real, then $f(z) = \overline{f(\overline{z})}$, therefore the complex roots are in pairs.

On $|z| = \frac{1}{4}$, $|g(z)| > \frac{1}{2}\frac{1}{4} = \frac{1}{8}$.

$$|f(z) - g(z)| < |a_0| + |a_2||\frac{1}{4}|^2 + \cdots |a_n||\frac{1}{4}|^n < \frac{1}{25} + \frac{1}{16}\frac{1}{1-\frac{1}{4}} = \frac{1}{25} + \frac{1}{12} = \frac{37}{300} < \frac{1}{8}.$$

The probability that $|a_1| > \frac{1}{2}$ and $|a_0| < \frac{1}{25}$ happens is 2%, and by Rouché theorem, f and g share the same number of roots in the circle $|z| = \frac{1}{4}$, which is one, thus that root must be real with the smallest modulus.

2 Contour Integrals

Problem 2.1.

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+4)(x^2+9)} dx$$

The upper semi-circle contour applies here.

$$\int_{-R}^{R} \frac{1}{(x^2+4)(x^2+9)} dx + \int_{C_R} \frac{1}{(z^2+4)(z^2+9)} dz$$
$$= 2\pi i(\mathrm{Res}(f, 2i) + \mathrm{Res}(f, 3i)) = 2\pi i(\frac{1}{20i} + \frac{1}{-30i}) = \frac{\pi}{30}.$$

Problem 2.2.

$$\int_0^{\infty} \frac{1}{1+x^n} dx, \quad n \geq 2$$

The contour follows the sector shape between $\theta = 0, \frac{2\pi}{n}$.

$$\int_0^R \frac{1}{1+x^n} dx + \int_{C_R} \frac{1}{1+z^n} dz - \int_0^R \frac{1}{1+x^n} e^{i2\pi/n} dx = 2\pi i \,\mathrm{Res}(f, e^{i\pi/n}) = \frac{2\pi i}{ne^{i\pi(n-1)/n}}$$

Then the integral is

$$\frac{2\pi i}{ne^{i\pi(n-1)/n}} \frac{1}{1-e^{i2\pi/n}} = \frac{\pi}{n\sin(\pi/n)}.$$

Problem 2.3.

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{1+x^2} dx, \quad a \in \mathbb{R}$$

Use the upper semi-circle, we assume $a > 0$,

$$\int_{-R}^{R} \frac{e^{iaz}}{1+z^2} dz + \int_{C_R} \frac{e^{iaz}}{1+z^2} dz = 2\pi i \,\mathrm{Res}(f, i) = 2\pi i e^{-a}/2i = \pi e^{-a}.$$

If $a<0$, we flip its sign. Therefore, the result is $\pi e^{-|a|}$.

Problem 2.4.

$$\int_0^\infty \cos(x^2)dx$$

Use the $\frac{\pi}{4}$ sector.

$$\int_0^\infty e^{ix^2}dx + \int_{C_R} e^{iz^2}dz - e^{i\frac{\pi}{4}} \int_0^\infty e^{i(i)x^2}dx = 0$$

Then the result is $\frac{\sqrt{2}}{2}\int_0^{-\infty} e^{-x^2}dx = \frac{\sqrt{2\pi}}{4}$.

Problem 2.5.

$$\int_0^\infty \frac{\sin^2 x}{x^2}dx$$

Take the semi-circle C_R and a tiny semi-circle C_ε around the origin as the contour. Write $\sin^2(z) = \frac{1-\cos(2z)}{2}$,

$$\int_\varepsilon^R \frac{1-e^{i2x}}{2x^2}dx + \int_{-R}^{-\varepsilon} \frac{1-e^{i2x}}{2x^2}dx + \int_{C_\varepsilon} \frac{1-e^{i2z}}{2z^2}dz + \int_{C_R} \frac{1-e^{i2z}}{2z^2}dz = 0$$

The first two terms $\to 2\int_0^\infty \frac{\sin^2 x}{x^2}dx$, the integral on C_R is negligible since $\mathrm{Im}(z) > 0$. The integral on C_ε means $z = \varepsilon e^{i\theta}$,

$$\int_{C_\varepsilon} \frac{1-e^{i2z}}{2z^2}dz = \int_0^\pi \frac{1-e^{i2\varepsilon e^{i\theta}}}{2\varepsilon^2 e^{i2\theta}}\varepsilon i e^{i\theta}d\theta = \int_0^\pi \frac{1-(1+i2\varepsilon e^{i\theta}+o(\varepsilon))}{2\varepsilon e^{i\theta}}id\theta$$

$$\to_{\varepsilon\to 0} \int_0^\pi 1d\theta = \pi$$

Thus, the result is $\frac{\pi}{2}$.