Lecture Notes for Math 5630/6630 **Fall 2024**

Note 7: Chebyshev Interpolation

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Disclaimer: *This lecture note is for math 5630/6630 class only.*

1 Chebyshev Interpolation

The Chebyshev interpolation aims to minimize the bound of the interpolation error. The bound of $\omega(x)$ only depends on the choice of the nodes, so a natural question is:

What kind of interpolation nodes will minimize $\max_{x\in[a,b]} \prod_{j=0}^{n} |x - x_j|$?

We first restrict our analysis to the interval $[a, b] = [-1, 1]$ for simplicity, the general case will be discussed later.

Example 1.1. *When $n = 1$, $\omega(x) = (x - x_0)(x - x_1)$, this function changes sign over the sub-intervals $[-1, x_0)$, (x_0, x_1), $(x_1, 1]$, then we can compute the maximum of $|\omega(x)|$ on these sub-intervals. Therefore we need to solve*

$$\min_{x_0,x_1\in[-1,1]} \max((1 + x_0)(1 + x_1), \frac{(x_1 - x_0)^2}{4}, (1 - x_0)(1 - x_1)),$$

while we can observe that

$$\frac{1}{2}(1 + x_0)(1 + x_1) + \frac{(x_1 - x_0)^2}{4} + \frac{1}{2}(1 - x_0)(1 - x_1) = 1 + \frac{(x_0 + x_1)^2}{4} \geq 1$$

holds for any choice of x_0, x_1, which means the maximum is at least $\frac{1}{2}$, it occurs when all terms are equal and $x_0 + x_1 = 0$. Hence $x_0 = -\frac{\sqrt{2}}{2} = \cos(3\pi/4), x_1 = \frac{\sqrt{2}}{2} = \cos(\pi/4)$.

The values $\pi/4$ and $3\pi/4$ are special, they are the roots of the function $\cos(2 \cdot x) = 0$.

Similarly, when $n = 0$, $\omega(x) = x - x_0$, then $\max|x - x_0| = \max(1 - x_0, x_0 + 1)$.

$$\min_{x_0} \max(1 - x_0, x_0 + 1) \geq \min_{x_0} \frac{1}{2}(1 - x_0 + x_0 + 1) = 1.$$

$x_0 = 0 = \cos(\pi/2)$ is the optimal solution. Here $\pi/2$ is the root of $\cos(1 \cdot x) = 0$.

1.1 Chebyshev Polynomials

Definition 1.2. *The Chebyshev polynomials of the* ***first kind*** *are defined by:*

$$T_n(x) = \cos(n \arccos x), \quad x \in [-1, 1].$$

Theorem 1.3. *The Chebyshev polynomial of the first kind satisfies the following:*

1. $T_n(\cos\theta) = \cos n\theta, \quad \theta \in [0, \pi]$.
2. $T_0 \equiv 1$, $T_1(x) = x$ *and* $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad n \geq 1$.
3. $\max_{x\in[-1,1]}|T_n(x)| = 1$.
4. *The leading coefficient of* $T_n(x)$ *is* 2^{n-1}.
5. T_n *has a total of* $(n+1)$ *extrema* $s_j = \cos(\frac{j\pi}{n}), j = 0, 1, \ldots, n$ *in the interval* $[-1, 1]$ *such that* $T_n(s_j) = (-1)^j$.

The 1st/3rd/5th claim is the definition.

The 2nd claim comes from the equality

$$\cos((n+1)\theta) + \cos((n-1)\theta) = 2\cos\theta\cos(n\theta)$$

The 4th claim can be derived by induction.

The Chebyshev polynomial of the first kind $T_{n+1}(x)$ have roots at

$$n \arccos x_k = \frac{\pi}{2} + k\pi \Rightarrow x_k = \cos(\frac{\pi}{2n} + \frac{k\pi}{n+1}), \quad k = 0, 1, \cdots, n.$$

Using the Fundamental Theorem of Algebra, we can write $T_{n+1}(x)$ as

$$T_{n+1}(x) = 2^n \prod_{k=0}^{n} (x - \cos(\frac{\pi}{2(n+1)} + \frac{k\pi}{n+1})).$$

Example 1.4. *If we take the nodes* $x_k = \cos(\frac{\pi}{2(n+1)} + \frac{k\pi}{n+1})$, *then*

$$\max_{x\in[-1,1]} \left| \prod_{k=0}^{n} (x - x_k) \right| = \max_{x\in[-1,1]} \frac{1}{2^n} |T_{n+1}(x)| = \frac{1}{2^n}.$$

The following theorem shows this bound is the smallest possible.

Theorem 1.5. *The optimal choice of interpolation nodes that minimize* $\max|\omega(x)|$ *is the extrema of Chebyshev polynomial* T_{n+1}.

$$\min_{x_j\in[-1,1]}\max_{x\in[-1,1]}|\omega(x)|=\max_{x\in[-1,1]}\frac{1}{2^n}|T_{n+1}(x)|=\frac{1}{2^n}$$

Let the roots of $T_{n+1}(x)$ be $z_0, z_1, \dots, z_n \in [-1,1]$, then we can write

$$T_{n+1} = 2^n(x-z_0)(x-z_1)\dots(x-z_n)$$

therefore $\frac{1}{2^n}T_{n+1}(x)$ is a polynomial with leading coefficient as 1. Since $\max_{x\in[-1,1]}|T_{n+1}(x)|=1$, it is clear that $\max_{x\in[-1,1]}\frac{1}{2^n}|T_{n+1}(x)|=\frac{1}{2^n}$. For the first equality, we try to prove it by contradiction. Let $x_0, x_1, \dots, x_n \in [-1,1]$, such that

$$\max_{x\in[-1,1]}|\omega(x)|<\frac{1}{2^n},$$

then we define the polynomial $\psi(x) = \frac{1}{2^n}T_{n+1}(x) - \omega(x)$, its degree is at most n due to cancellation, therefore at most have n zeros. On the other hand, because $\frac{1}{2^n}T_{n+1}(s_j) = \frac{1}{2^n}(-1)^j$ at the extrema $s_j = \cos(\frac{j\pi}{n+1})$, $j = 0, \dots, (n+1)$, the polynomial $\psi(s_j)$ must share the same sign of $\frac{1}{2^n}T_{n+1}(s_j)$. This means $\psi(x)$ changes sign $(n+1)$ times, hence $(n+1)$ zeros. It is a contradiction.

Definition 1.6. *The interpolation nodes* $z_j = \cos(\frac{(2j+1)\pi}{2(n+1)})$, $j = 0, 1, \dots, n$ *are called* ***Chebyshev nodes****. These nodes are the zeros of Chebyshev polynomial* T_{n+1}.

As a comparison with Chebyshev nodes, if $x_k = -1 + \frac{2k}{n}$, $k = 0, \cdots, n$ are the equally spaced nodes, the estimate of (by Stirling Formula)

$$\max_{x\in[-1,1]}\left|\prod_{k=0}^{n}(x-x_k)\right| \le \frac{n!}{4}\frac{2^{n+1}}{n^{n+1}} \approx \frac{\sqrt{2\pi}}{2\sqrt{n}}\left(\frac{2}{e}\right)^n.$$

This term decays slower than 2^{-n}.

Now we can generalize the Chebyshev nodes to interval $[a,b]$. One can defined the affine transformation ϕ mapping $[-1,1]$ to $[a,b]$ by $\phi(x) = \frac{1}{2}(a+b+(b-a)x)$. See Runge's example with Chebyshev nodes in Figure **??** (in comparison with the figure of the last note).

It is not difficult to prove the following.

Corollary 1.7. *The optimal choice of interpolation nodes that minimize* $\max|\omega(x)|$ *on* $[a,b]$ *are* $\phi(z_j)$ *and*

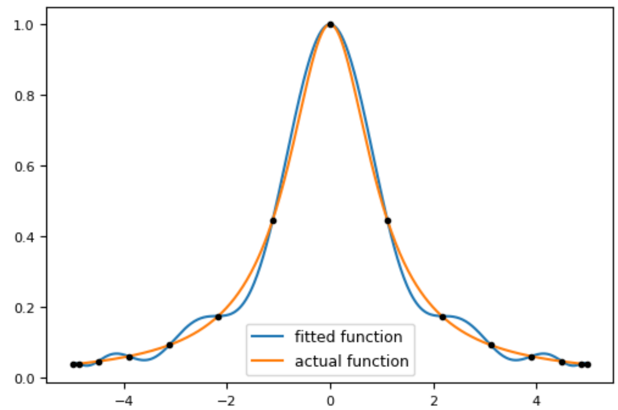

Figure 7.1: Interpolation Runge's function at Chebyshev nodes.

$$\min_{x_j\in[a,b]}\max_{x\in[a,b]}|\omega(x)|=\frac{(b-a)^{n+1}}{2\cdot 4^n}.$$

This bound is much smaller than the bound for equally spaced nodes.

1.2 Stability of Interpolation

The resulting polynomial interpolation will have an error when the data points are inaccurate. Suppose the data points are $(x_0, y_0+\varepsilon_0)$, $(x_1, y_1+\varepsilon_1)$, $\cdots$, $(x_n, y_n+\varepsilon_n)$, where $\{\varepsilon_k\}_{k=0}^n$ are noises. Suppose p_n and $\widetilde{p}_n$ are the interpolating polynomials for clean and noisy data points, then $p_n-\widetilde{p}_n$ is the polynomial interpolating on data points $(x_0,\varepsilon_0),\cdots,(x_n,\varepsilon_n)$. Therefore, using Lagrange polynomials we can represent

$$p_n(x)-\widetilde{p}_n(x)=\sum_{k=0}^n\varepsilon_k L_k(x),$$

where $L_k(x)$ are the Lagrange polynomials. Furthermore,

$$\max_{x\in[a,b]}|p_n(x)-\widetilde{p}_n(x)|\le\max_{x\in[a,b]}\sum_{k=0}|\varepsilon_k||L_k(x)|\le\left(\max_{0\le k\le n}|\varepsilon_k|\right)\left(\max_{x\in[a,b]}\sum_{k=0}^n|L_k(x)|\right).$$

The function $\lambda(x):=\sum_{k=0}^n|L_k(x)|$ is the Lebesgue function. It is a piecewise polynomial. The quantity $\Lambda=\max_{x\in[a,b]}\sum_{k=0}^n|L_k(x)|$ is the **Lebesgue constant**. This constant determines the stability of polynomial interpolation.

On equally spaced nodes, this constant is quite large.

Theorem 1.8. *Let $\{x_j\}_{j=0}^n$ be equally spaced nodes, then the Lebesgue constant*

$$\Lambda\approx\frac{2^{n+1}}{en\ln n}$$

It is proved by Paul Erdös in 1964 that for any set of interpolation nodes $\{x_j\}_{j=0}^n$

$$\Lambda > \frac{2}{\pi}\log(n+1) + \frac{1}{2}.$$

As the number of nodes $n \to \infty$, this constant $\Lambda \to \infty$. This leads to the result of Faber:

> For any choice of nodes, there exists a continuous function not able to be approximated by the interpolating polynomial.

The Chebyshev nodes are almost optimal in the sense that

$$\Lambda_{Chebyshev} < \frac{2}{\pi}\log(n+1) + 1.$$

The set of nodes that minimizes Λ is generally difficult to compute. There exists a slightly better set of nodes than Chebyshev nodes, *extended Chebyshev nodes*,

$$\widetilde{x}_j = \frac{\cos(\frac{2j+1}{2(n+1)}\pi)}{\cos(\frac{\pi}{2(n+1)})}.$$

Compare that to the Chebyshev nodes $x_j = \cos(\frac{2j+1}{2(n+1)}\pi)$.