Lecture Notes for Math 5630/6630 **Fall 2024**

Note 1: Review of Essentials

Tags: `math.na` *Date: 08/22/2024*

Disclaimer: *This lecture note is for math 5630/6630 class only.*

1 Useful Theorems

Theorem 1.1 (Intermediate Value Theorem)**.** *Suppose f is continuous on $[a, b]$. If $f(a)f(b) < 0$, then f has a root on $[a, b]$.*

Example 1.2. *The function $f(x) = e^x - x - 2$ has a root on $[0, 2]$.*

Since $f(0) = -1 < 0$ and $f(2) = e^2 - 4 = (e - 2)(e + 2) > 0$.

Theorem 1.3 (Mean-Value Theorem)**.** *There is a number $c \in [a, b]$ that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

We can create a function $g(x) = f(a) + (x - a)\frac{f(b)-f(a)}{b-a}$, then $g(x) - f(x) = 0$ at $x = a$ and $x = b$. Rolle's theorem implies there is a number $c \in [a, b]$ that $g'(c) = f'(c)$.

The mean-value theorem implies the following consequence.

Theorem 1.4. *There is a number $\zeta \in [a, b]$ that*

$$\int_a^b h(t)dt = (b - a)h(\zeta).$$

Take $f(x) = \displaystyle\int_0^x h(t)dt$ into the mean-value theorem.

2 Taylor series

The Taylor series of a smooth function $f(x)$ at x_0 is

$$f(x) \sim f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \cdots \tag{0.1}$$

Example 2.1. *Find the Taylor series of $f(x) = e^{-x}$ at $x = \ln(2)$.*

The derivatives $\frac{d^k}{dx^k} e^{-x} = (-1)^k e^{-x}$, the coefficient before $(x - x_0)^k$ is $\frac{(-1)^k}{2k!}$, then the Taylor series is

$$\frac{1}{2} + \frac{-1}{2}(x - \ln 2) + \frac{1}{2}\frac{1}{2!}(x - \ln 2)^2 + \frac{-1}{2}\frac{1}{3!}(x - \ln 2)^3 + \cdots$$

Example 2.2. *Find the first 2 nonzero terms of Taylor series of $f(x) = \tan(x)$ at $x = 0$.*

The kth coefficient is $\frac{1}{k!}\frac{d^k}{dx^k} f(x)|_{x=x_0}$,

$$f(0) = \tan(x)\Big|_{x=0} = 0, \quad f'(0) = \frac{1}{\cos^2(x)}\Big|_{x=0} = 1, \quad f''(0) = \frac{2\sin(x)}{\cos^3(x)}\Big|_{x=0} = 0$$

$$f'''(0) = \frac{2 + 4\sin^2(x)}{\cos^4(x)}\Big|_{x=0} = 2.$$

Therefore

$$\tan(x) \sim x + \frac{1}{3}x^3 + \cdots$$

The Taylor series at $x = x_0$ can be used as an approximation to $f(x)$ when $|x - x_0|$ is small. The error of approximation can be estimated using the <u>remainder formula</u>.

Theorem 2.3.

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{f^{(m)}(x_0)}{m!}(x - x_0)^m + \frac{f^{(m+1)}(\zeta)}{(m+1)!}(x - x_0)^{m+1}. \tag{0.2}$$

where ζ is between x and x_0.

The derivation is simple if we notice that

$$g(x) = f(x) - \left(f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{f^{(m)}(x_0)}{m!}(x - x_0)^m \right)$$

satisfies $g(x_0) = g'(x_0) = g''(x_0) = \cdots = g^{(m)}(x_0) = 0$. Finally, use integration by parts,

$$\begin{aligned}
\int_{x_0}^{x} g^{(m+1)}(t)\frac{(x-t)^m}{m!}dt &= \cancel{g^{(m)}(t)\frac{(x-t)^m}{m!}\Big|_{x_0}^{x}} + \int_{x_0}^{x} g^{(m)}(t)\frac{(x-t)^{m-1}}{(m-1)!}dt \\
&= \cancel{g^{(m-1)}(t)\frac{(x-t)^{m-1}}{(m-1)!}\Big|_{x_0}^{x}} + \int_{x_0}^{x} g^{(m-1)}(t)\frac{(x-t)^{m-2}}{(m-2)!}dt \\
&= \cdots = \int_{x_0}^{x} g'(t)dt = g(x) - g(x_0) = g(x).
\end{aligned}$$

The mean-value theorem implies that there is a $\zeta \in [x_0, x]$ (or $[x, x_0]$)

$$\int_{x_0}^{x} g^{(m+1)}(t)\frac{(x-t)^m}{m!}dt = g^{(m+1)}(\zeta)\int_{x_0}^{x} \frac{(x-t)^m}{m!}dt = g^{(m+1)}(\zeta)\frac{(x-x_0)^{m+1}}{(m+1)!}.$$

The remainder estimates the error of the approximation by Taylor series.

Theorem 2.4. *For $x \in D = [x_0 - R, x_0 + R]$, the approximation error is bounded by*

$$\left| f(x) - \left(f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{f^{(m)}(x_0)}{m!}(x - x_0)^m \right) \right| \le \max_{\zeta \in D} |f^{(m+1)}(\zeta)| \frac{R^{m+1}}{(m+1)!}.$$

Example 2.5. *Let $f(x) = e^x$, $x \in D = [-1, 1]$. Estimate the approximation error of the Taylor series with 11 terms centered at $x_0 = 0$.*

The approximation error of Taylor series

$$f_n(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

is bounded by $\frac{e}{(n+1)!}$. Truncation at $n = 10$ will give a bound of 6.81×10^{-8} on D.

Remark 2.6. *It should be noted that the bound*

$$\max_{\zeta \in D} |f^{(m+1)}(\zeta)| \frac{R^{m+1}}{(m+1)!}$$

may not converge to zero as $m \to \infty$. The derivative can grow very quickly in m.

3 Ordinary Differential Equations

The 1st order ODE has the form

$$y' = f(t, y)$$

with initial condition $y(t_0) = y_0$. A higher-order ODE can be converted into 1st order ODE system. For instance,

$$y^{(n)} = f(t, y, y', \cdots, y^{(n-1)}).$$

can be written as a system of n functions.

$$\begin{aligned}
z_1'(t) &= z_2(t), \\
z_2'(t) &= z_3(t), \\
&\cdots \\
z_{n-1}'(t) &= z_n(t), \\
z_n'(t) &= f(t, z_1(t), z_2(t), \cdots, z_n(t)).
\end{aligned}$$

We can write the above equation into a general vector form

$$\frac{d}{dt}\mathbf{Z}(t) = \mathbf{F}(t, \mathbf{Z}(t)),$$

where $\mathbf{Z} = (z_1, \cdots, z_n)$.

Theorem 3.1 (Picard–Lindelöf theorem)**.** *Let $D \subset \mathbb{R} \times \mathbb{R}^n$ be a closed rectangle and $(t_0, \mathbf{Z}_0)$ is an interior point of D, if $\mathbf{F} : D \mapsto \mathbb{R}^n$ is a continuous function in t and Lipschitz continuous in $\mathbf{Z}$, then there exists $\varepsilon > 0$ that the ODE*

$$\frac{d}{dt}\mathbf{Z}(t) = \mathbf{F}(t, \mathbf{Z}(t)), \quad \mathbf{Z}(t_0) = \mathbf{Z}_0$$

has a unique local solution $\mathbf{Z}(t)$ on $[t_0 - \varepsilon, t_0 + \varepsilon]$.

Example 3.2. *The ODE*

$$\frac{d^2y}{dt^2} = ty^2 + (y')^2$$

has a unique local solution around $t = 0$ for each initial condition $y(0) = a$, $y'(0) = b$.

Because the function $\mathbf{F}(t, z_1, z_2) = (z_2, tz_1^2 + z_2^2)$ is continuous over the whole space and $\nabla_{z_1}\mathbf{F}$, $\nabla_{z_2}\mathbf{F}$ are both continuous. According to the Picard-Lindelof theorem, the solution exists in a small rectangle D around the initial point.

It cannot be extended to global uniqueness since the function $\mathbf{F}$ may not be Lipschitz continuous anymore. For instance

$$y' = y^2, \quad y(0) = y_0,$$

has a solution $y(t) = \frac{y_0}{1 - y_0 t}$, which blows up at $t = \frac{1}{y_0}$. Although $f(y) = y^2$ is continuous, f is not Lipschitz continuous since $f'(y) = 2y$ has no bound on $y \in \mathbb{R}$.

4 Bachmann–Landau Notations

The Bachmann–Landau notations characterize the limiting behavior as the argument goes to infinity.

4.1 Big-O

Definition 4.1. *We write $f(x) = O(g(x))$ as $x \to \infty$, if there is a constant M and a number x_0 that $|f(x)| \leq Mg(x)$ for $x > x_0$.*

Example 4.2. *If $f(x) = \sin x$ and $g(x) = 1$, then $f(x) = O(g(x))$ as $x \to \infty$.*

Example 4.3. *If $f(x) = x + 1$, $g(x) = x^2$, then $f(x) = O(g(x))$ as $x \to \infty$.*

Example 4.4. *If $f(x) = \log x$, $g(x) = \sqrt{x}$, then $f(x) = O(g(x))$ as $x \to \infty$.*

Example 4.5. *If $f(n) = n \log n$ and $g(n) = \log(n!)$, then $f(n) = O(g(n))$ as $x \to \infty$.*

4.2 Big-Omega

Definition 4.6 (Knuth)**.** *We write $f(x) = \Omega(g(x))$ as $x \to \infty$, if $g(x) = O(f(x))$.*

Example 4.7. *If $f(x) = x^{1/2}$, $g(x) = x^{1/3}$, then $f(x) = \Omega(g(x))$ as $x \to \infty$.*

Example 4.8. *If $f(n) = n^2$ and $g(n) = n \log n$, then $f(n) = \Omega(g(n))$ as $n \to \infty$.*

4.3 Big Theta

The big-theta notation is the combination of big-O and big-Omega.

Definition 4.9. *We write $f(x) = \Theta(g(x))$ as $x \to \infty$, if $f(x) = O(g(x))$ and $f(x) = \Omega(g(x))$. This implies that $f(x)$ and $g(x)$ are comparable.*

Example 4.10. *If $f(x) = x^2$, $g(x) = 2x^2 + x + 1$, then $f(x) = \Theta(g(x))$ as $x \to \infty$.*

The above definitions can be extended to limiting behavior $x \to a$ instead of infinity. For instance, $h^2 = O(h)$ as $h \to 0$ which shows h is relatively larger than h^2 as h approaches zero.