Math 5630/6630 **Fall 2024**

Homework 7

Tags: Runge Kutta *Due Date: 12/11/2024 11:59PM CST*

1 Homework Problems

This part of the homework assignment should be submitted via Canvas. You can scan the answers into PDF files, or typeset them in Word or LaTeX, then convert/compile them into PDF files.

Problem 1.1. *Use the Taylor series to show that the forward Euler method*

$$y_{n+1} = y_n + hf(t_n, y_n)$$

has an error of $\mathcal{O}(h)$ for the local truncation error.

Problem 1.2. *Use the Taylor series to show that the midpoint Euler method*

$$y_{n+1} = y_n + hf(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_1), \quad k_1 = f(t_n, y_n)$$

has an error of $\mathcal{O}(h^2)$ for the local truncation error. You will need

$$y''(t) = \frac{d}{dt}f(t,y) = \partial_t f(t,y) + \partial_y f(t,y)y'(t) = \partial_t f(t,y) + \partial_y f(t,y)f(t,y).$$

1.1 Extra Problems for MATH 6630

Problem 1.3. *As a generalization of Problem 1.2, show that*

$$y_{n+1} = y_n + h(b_1k_1 + b_2k_2), \quad k_1 = f(t_n, y_n), \quad k_2 = f(t_n + a_2h, y_n + c_2hk_1)$$

can achieve $\mathcal{O}(h^2)$ local truncation error if

$$b_1 + b_2 = 1, \quad b_2a_2 = \frac{1}{2}, \quad b_2c_2 = \frac{1}{2}.$$

2 Programming Assignments

Implement the following program tasks using your favorite programming language. The `Python` or `MATLAB` starter kit is available at GitHub Link. Follow the guidelines there for your submission.

Problem 2.1. *Implement three methods to solve the ODE $y' = f(t, y)$ with initial condition $y(0) = y_0$.*

(a) Forward Euler: $y_{n+1} = y_n + hf(t_n, y_n)$.

(b) Midpoint Euler: $y_{n+1} = y_n + hf(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_1)$, $k_1 = f(t_n, y_n)$

(c) 4th order Runge Kutta:

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where

$$\begin{aligned} k_1 &= f(t_n, y_n), \\ k_2 &= f(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_1), \\ k_3 &= f(t_n + \frac{h}{2}, y_n - \frac{h}{2}k_2), \\ k_4 &= f(t_n + h, y_n + hk_3). \end{aligned}$$

Problem 2.2. *Use your Runge Kutta method (`RK4`) to solve the following ODE*

$$y' = t(y - t\sin t), \quad y(0) = 1$$

over the domain $t \in [0, 10]$.

i. Plot your numerical solutions (with different h*) along with the exact solution* $y_{exact} = t\sin t + \cos(t)$.

ii. Does the numerical error satisfy the theoretical estimate $\mathcal{O}(h^4)$*? Why or why not? Answer that with comments in the submission script.*

2.1 Extra Problems for MATH 6630

Problem 2.3. *There are several 4th-order Runge Kutta methods. Besides the usual `RK4` method, there is another 4th-order method called* $\frac{3}{8}$*-rule (shown as the following formula).*

$$y_{n+1} = y_n + \frac{h}{8}\left(k_1 + 3k_2 + 3k_3 + k_4\right)$$

where

$$\begin{aligned}
k_1 &= f(t_n, y_n),\\
k_2 &= f(t_n + \frac{h}{3}, y_n + \frac{h}{3}k_1),\\
k_3 &= f(t_n + \frac{2h}{3}, y_n - \frac{h}{3}k_1 + hk_2),\\
k_4 &= f(t_n + h, y_n + hk_1 - hk_2 + hk_3).
\end{aligned}$$

Implement this $\frac{3}{8}$*-rule and test with the Problem 2.2. Compare* $\frac{3}{8}$*-rule with `RK4` in total running time and the global errors, and put your conclusions in the comments.*