Math 5630/6630 **Fall 2024**

Homework 6

Tags: Quadrature *Due Date: 11/29/2024 11:59PM CST*

1 Homework Problems

This part of the homework assignment should be submitted via Canvas. You can scan the answers into PDF files, or typeset them in Word or LaTeX, then convert/compile them into PDF files.

Problem 1.1. *Show the quadrature rule*

$$\int_0^1 f(x)dx \approx \frac{1}{6}f(0) + \frac{2}{3}f\left(\frac{1}{2}\right) + \frac{1}{6}f(1)$$

is accurate for all polynomials of degree ≤ 3.

Problem 1.2. *Use* ***Lagrange interpolation*** *to construct a quadrature rule on* $[0, 1]$ *with nodes*

$$x_0 = \frac{1}{2}, \quad x_1 = \frac{3}{4}, \quad x_2 = 1.$$

Is the quadrature rule accurate for $f(x) = x^3$*?*

Problem 1.3. *Solve a linear system to derive a quadrature rule*

$$\int_0^1 f(x)dx \approx w_0 f(0) + w_1 f(\frac{1}{3}) + w_2 f(\frac{4}{5}) + w_3 f(1)$$

Find the degree of accuracy of this quadrature rule (Do not round the coefficients, leave them in rational numbers, and be patient).

Problem 1.4. *Let* $\{x_k\}$, $k = 0, 1, 2, \cdots, n$ *be equally spaced nodes on* $[-1, 1]$.

1. *Let* $p_n(x) = \sum_{k=0}^{n} f(x_k)L_k(x)$ *be the interpolation polynomial for* $f(x)$ *at the nodes. Use the* ***error estimate of interpolation*** *to find the error of the quadrature rule*

$$\int_{-1}^1 f(x)dx \approx \sum_{k=0}^{n} w_k f(x_k), \quad w_k = \int_{-1}^1 L_k(x)dx.$$

2. *According to the above derivation process for error estimate, do you think Chebyshev nodes (same number of nodes) will be more accurate or not? Explain your choice.*
3. *Explain why the quadrature rule is accurate for all polynomials with degree* $\leq n$.

Problem 1.5. *We construct a simple Gauss quadrature rule.*

1. *Find polynomials* $p_k(x)$ *of degree* k, $k = 0, 1, 2$ *such that*

$$\int_{-1}^1 p_i(x)p_j(x)dx = 0, \quad i \neq j$$

2. *Let* x_0, x_1 *be the roots of* $p_2(x)$*, find a quadrature rule for*

$$\int_{-1}^{1} f(x)dx \approx w_0 f(x_0) + w_1 f(x_1).$$

   What is the degree of accuracy of this quadrature rule?

1.1 Extra Problems for MATH 6630

Definition 1.6. *Two polynomials* $A(x)$ *and* $B(x)$ *are* ***orthogonal*** *on* $[a, b]$ *if* $\int_a^b A(x)B(x)dx = 0$.

Problem 1.7. *As a follow-up for Problem 1.3. Suppose you have the freedom to choose two other points* $a, b \in (0, 1)$ *to construct the quadrature rule*

$$\int_0^1 f(x)dx \approx w_0 f(0) + w_1 f(a) + w_2 f(b) + w_3 f(1).$$

To maximize the degree of accuracy, you need to make the polynomial $p(x) = (x - 0)(x - a)(x - b)(x - 1)$ *to be* ***orthogonal*** *to* $f(x) = 1$ *and* $g(x) = x$.

1. *When* $a = \frac{1}{3}$ *and* $b = \frac{4}{5}$*, show* $p(x)$ *is orthogonal to* 1 *only.*
2. *Find* a, b*, what is the degree of accuracy for the resulting quadrature rule? (You may change the variables to solve it, say* $u = a + b$ *and* $v = ab$*)*

2 Programming Problems

Implement the following program tasks using your favorite programming language. The `Python` or `MATLAB` starter kit is available at GitHub Link. Follow the guidelines there for your submission.

Problem 2.1. *Implement the composite quadrature rules with choices of the midpoint, trapezoidal, and Simpson rules on each subinterval.*

1. *On each subinterval* $[x_k, x_{k+1}]$*, the midpoint rule is* $hf(\frac{x_k+x_{k+1}}{2})$.
2. *On each subinterval* $[x_k, x_{k+1}]$*, the trapezoidal rule is* $(h/2)\cdot(f(x_k)+f(x_{k+1}))$.
3. *Assuming the total number of subintervals is even. On the interval* $[x_k, x_{k+2}]$ *of size* $2h$*, the Simpson's rule is* $\frac{2h}{6}(f(x_k)+4f(x_{k+1})+f(x_{k+2}))$.

Problem 2.2. *Implement Problem* **??** *first, then run the* `hw06.p2()` *(MATLAB) or* `p2()` *(Python) function in the script to generate some plots about the convergence and make comments about the plots.*

It will use your implemented composite methods in Problem **??** *to integrate a few example functions on* $[-1,1]$*. The example functions are:*

$$f_1(x) = e^x, \quad f_2(x) = (1-x^2)^3, \quad f_3(x) = (1-x^2)^5, \quad f_4(x) = (1-x^2)^7$$

The key difference between f_1 *and the others is that* f_2, f_3, f_4 *functions have their first a few derivatives at* $x=-1$ *and* $x=1$ *equal. But* f_1 *does not have that property. You will find out how this property can affect the convergence rate.*

Problem 2.3. *Implement the Romberg method with your codes for Problem* **??** *and Richardson extrapolation (in HW05).*

The composite rules satisfy

$$I(h) = I(0) + C_1h^2 + C_2h^4 + \cdots$$

if using midpoint or trapezoidal rules. For Simpson's rule,

$$I(h) = I(0) + D_1 h^4 + D_2 h^6 + \cdots$$

Problem 2.4. *Follow the instructions in the script to construct the Gauss quadrature rule with 6 nodes.*

1. *Compute the roots (nodes) of Legendre polynomial* $P_6(x) = \frac{1}{16}\left(231x^6 - 315x^4 + 105x^2 - 5\right)$.

   There are 6 roots are on brackets $(-1, -\frac{3}{4})$, $(-\frac{3}{4}, -\frac{1}{4})$, $(-\frac{1}{4}, 0)$, $(0, \frac{1}{4})$, $(\frac{1}{4}, \frac{3}{4})$, $(\frac{3}{4}, 1)$, *respectively. Use either your bisection or false position codes (in HW02) to solve this equation, with a stopping criterion* $|P_6(x_n)| < 10^{-14}$.

2. *Compute the coefficients for these nodes by solving a linear system.*

2.1 Extra Problems for MATH 6630

Problem 2.5. *Construct the Gauss quadrature rule* $\{(w_k, x_k)\}$, $k = 1, \cdots, n$. *This will make use of your root-finding code.*

You need to first find the nodes $\{x_k\}_{k=1}^n$ *on* $(-1, 1)$, *they are the roots of the Legendre polynomial* $P_n(x)$ *of degree* n.

These nodes are very close to the values $\cos(\frac{4k-1}{4n+2}\pi)$, $k = 1, 2, \cdots, n$. *You may use them as the starting points and use Newton's method to find the roots.*

The codes of Legendre polynomial `legendre_poly(n, x)` *and its derivative* `deriv_lengendre_poly(n, x)` *are given in the script.*