Math 5630/6630 **Fall 2024**

Homework 3

Tags: *Interpolation (I)* *Due Date: 10/17/2024 11:59PM CST*

1 Homework Problems

This part of the homework assignment should be submitted via Canvas. You can scan the answers into PDF files, or typeset them in Word or LaTeX, then convert/compile them into PDF files.

Problem 1.1. *Find a cubic polynomial passing through* $(-3, 3)$, $(-1, 1)$, $(1, 1)$, $(3, 3)$. *Represent it using Lagrange polynomials.*

Problem 1.2. *Let* $f(x) = \sin(\pi x)$. *Estimate the error using polynomial interpolation at Chebyshev nodes on* $[-1, 1]$. *What is your estimated minimal degree to achieve an error of* 10^{-6}*?*

Problem 1.3. *Let* $f(x) = \sin(\pi x)$. *Estimate the error using polynomial interpolation at equally spaced nodes on* $[-1, 1]$. *What is your estimated minimal degree to achieve an error of* 10^{-6}*?*

1.1 Extra Problems for MATH 6630

Problem 1.4. *Let $\mathcal{X} = \{x_0, x_1, \cdots, x_n\}$ be a set of distinct real numbers, and let $L_j(x)$ be the Lagrange polynomials on $\mathcal{X}$, $j = 0, 1, \cdots, n$. Suppose $f(x)$ is a polynomial of degree n. Use the uniqueness of polynomial interpolation to explain why*

$$f(x) = \sum_{j=0}^{n} f(x_j) L_j(x).$$

Problem 1.5. *The Chebyshev nodes are the roots of Chebyshev polynomials of 1st kind on $[-1, 1]$. These nodes do not include the endpoints ± 1. To include the endpoints in the interpolation nodes, we can consider the Chebyshev extreme nodes,*

$$z_k = \cos\left(\frac{k}{n}\pi\right), \quad k = 0, \cdots, n.$$

i. *Let Chebyshev polynomial $T_n(x) := \cos(n \arccos x)$. Show that*

$$\frac{d}{dx} T_n(x) = \frac{n \sin(n \arccos x)}{\sin(\arccos x)},$$

and $\frac{d}{dx} T_n(x)$ is a polynomial of degree $(n-1)$ of leading coefficient $n 2^{n-1}$.

ii. *Show that $\frac{d}{dx} T_n(z_k) = 0$, $k = 1, \cdots, n-1$.*

iii. *The "Fundamental Theorem of Algebra" implies*

> *Theorem: If $p(x)$ is a polynomial of degree n with leading coefficient 1 and roots x_k, $k = 0, \cdots, n-1$, then*
>
> $$p(x) = \prod_{i=0}^{n-1} (x - x_k).$$

Use this theorem and (i), (ii) to derive

$$\prod_{k=0}^{n} (\cos\theta - z_k) = \frac{-\sin\theta \sin(n\theta)}{2^{n-1}}.$$

iv. *Show that the Chebyshev extreme nodes satisfy*

$$\max_{x \in [-1,1]} \prod_{k=0}^{n} |x - z_k| \leq \frac{1}{2^{n-1}}.$$

It is slightly worse than the Chebyshev nodes but the endpoints are included.

2 Programming Problems

Implement the following program tasks using your favorite programming language. The `Python` or `MATLAB` starter kit is available at GitHub Link. Follow the guidelines there for your submission.

Problem 2.1. *Implement Lagrange polynomial interpolation.*

Problem 2.2. *Use equally spaced nodes and Chebyshev nodes to compute the Lagrange polynomial interpolation for* $f(x) = \frac{1}{1+25x^2}$ *and* $g(x) = \sin(\pi x)$ *on* $[-1, 1]$. *Record the interpolation errors for different degrees. Comment on the interpolation error's trend as the degree increases (especially for equally spaced nodes).*

2.1 Extra Problems for MATH 6630

Problem 2.3. *Repeat the Problem 2.2 with the Chebyshev extreme nodes (see Problem 1.5) and Comment on the results compared with the Chebyshev nodes.*